A simple quasi-planar drawing of K_{10} ⋆

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Abstract. We show that the complete graph on ten vertices K_{10} is a simple quasi-planar graph, which answers a question of Ackerman and Tardos [E. Ackerman and G. Tardos, On the maximum number of edges in quasi-planar graphs, J. Comb. Theory, Series A 114 (2007) 563-571] and shows that the bound $6.5n - 20$ on the maximum number of edges of simple quasi-planar graphs is tight for $n = 10$.

A graph is *k-quasi-planar* if it can be drawn in the plane so that there are no k pairwise crossing edges. A k-quasi-planar graph is *simple* if each pair of edges meets at most once, either at a common endpoint or at a crossing point, and is *geometric* if each edge is drawn as a straight line segment. 3-quasi-planar graphs are called *quasi-planar*. Quasi-planar graphs were introduced by Agarwal et al. [2] and have intensively been studied since then, with a focus on Turán-type problems. Do k-quasi-planar graphs have at most a linear number of edges? Which is the maximum complete quasi-planar graph?

Ackerman and Tardos [1] proved that simple quasi-planar graphs have at most $6.5n - 20$ edges and this bound is tight up to a constant. There are no simple quasi-planar graphs with exactly $6.5n - 20$ edges. In consequence, K_{11} is not a simple quasi-planar graph. It is known that K_9 is a quasi-planar geometric graph whereas K_{10} is not a quasi-planar geometric graph [3]. Here, we draw K_{10} so that each pair of edges meets at most once and three edges do not pairwise cross, and thereby solve the open problem on K_{10}.

We partition K_{10} into two K_5 which are each drawn as an encircled pentagram (with solid black and dotted blue edges). Let $U = \{u_1, \ldots, u_5\}$ ($V = \{v_1, \ldots, v_5\}$) be the vertices of the outer (inner) K_5 in clockwise order. The remaining $K_{5,5}$ is drawn so that the edges $\{u_i, v_{i-1}\}, \{u_i, v_i\}$ and $\{u_i, v_{i+1}\}$ in circular order are drawn straight-line. Finally, (pink) edges $\{u_i, v_{i+2}\}$ go around u_{i+1} in the outer face, and (red) edges $\{u_i, v_{i-2}\}$ traverse the inner circle. The drawing is simple quasi-planar, since no three edges cross pairwise. For an inspection, first observe that the (dashed black) edges $\{u_i, v_i\}$ and the (black) edges on the circles of U and V are crossed at most once. Fig. 2(a) shows K_{10} after their removal. Thereafter, the edges $\{v_i, u_{i+1}\}$ (thick and green) are crossed by two edges which do not cross pairwise, and similarly for $\{v_i, u_{i-1}\}$ (cyan). Finally, in the remainder, each red edge crosses a pink, two red and two dotted blue edges, which do not cross mutually, and similarly for the pink edges. Alternatively, consider the edge intersection graph which is triangle-free.

⋆ Supported by the Deutsche Forschungsgemeinschaft (DFG), grant Br835/18-2.

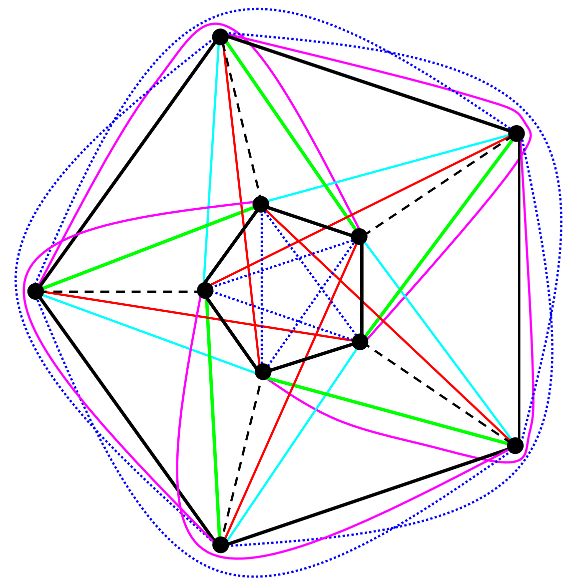
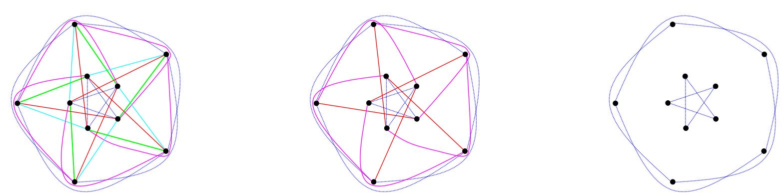


Fig. 1. A quasi-planar drawing of K_{10}



(a) (b) (c)

Fig. 2. Inspection of K_{10} after the removal of (a) black edges, (b) green and cyan edges, and (c) red and pink edges

References

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3. O. Aichholzer and H. Krasser. The point set order type data base: A collection of applications and results. In *Proc. 13th CCCG, Waterloo, Ontario, Canada, 2001*, pages 17–20, 2001.