A Sparse Stress Model

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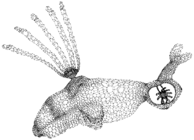
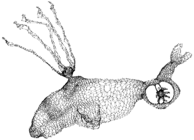
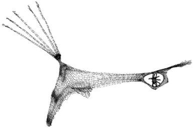


full stress (340.10s)



our approach (2.56s)



best related (3.60s)

24th Intl. Symp. Graph Drawing
19–21 September 2016, Athens

Full Stress Model

Goal:

- finding layout $x = (\mathcal{R}^2)^V$ minimizing:

$$s(x) \quad = \quad \sum_{i,j \in \binom{V}{2}} w_{ij}(||x_i - x_j|| - d_{ij})^2$$

with

- d_{ij} := shortest path distances between i and j
- $w_{ij} := 1/d_{ij}^2$

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- finding layout $x = (\mathcal{R}^2)^V$ minimizing:

$$
\begin{aligned}
s(x) &= \sum_{i,j \in \binom{V}{2}} w_{ij}(||x_i - x_j|| - d_{ij})^2 \\
&= \underbrace{\sum_{i,j \in E} w_{ij}(||x_i - x_j|| - d_{ij})^2}_{1^{\text{st}}\text{ term}} + \underbrace{\sum_{i,j \notin E} w_{ij}(||x_i - x_j|| - d_{ij})^2}_{2^{\text{nd}}\text{ term}}
\end{aligned}
$$

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- d_{ij} := shortest path distances between i and j
- $w_{ij} := 1/d_{ij}^2$

Complexity due to 2^{nd} term:

- preprocessing: $\mathcal{O}(n(m + n \log n))$
- time & space: $\mathcal{O}(n^2)$

How to deal with the 2nd Term

Related Sparse Stress Models:

- ignore:
  - GRIP (Gajer, Goodrich and Kobourov, *GD'00*, 2001)
  - 1-stress (Brandes and Pich, *GD'08*, 2009)
- replace:
  - maxent (Gansner, Hu and North, *TVCG*, 2013)
  - COAST (Gansner, Hu and Krishnan, *GD'13*, 2013)
- approximate shortest path matrix & set $w_{ij} = 1/d_{ij}$:
  - MARS (Khoury, Hu, Krishnan and Scheidegger, *CGF*, 2012)

our approach:

- approximate by term aggregation

Aggregation of Terms is an old Hat (or was it Hut)

Aggretation of terms via Barnes & Hut: (Barnes and Hut, *Nature*, 1986)

1. construct partition of V for each $i \in V$
2. for each partition $p \in \mathcal{P}(i)$ determine centroid
3. **aggregation:** repulsive forces only w.r.t. centroids

Result:

- repulsive forces: $\mathcal{O}(n \log n)$ instead of $\mathcal{O}(n^2)$ computations

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Problem:

- centroids artificial nodes $\Rightarrow$ no shortest-path distance

A Sparse Stress Model

Solution:

- select k pivots $\mathcal{P} \subseteq V$
- assign each pivot p a partition $\mathcal{R}(p) \subseteq V$ with $p \in \mathcal{R}(p)$ and $\mathcal{R}(p) \cap \mathcal{R}(p') = \emptyset$

A Sparse Stress Model

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Our Sparse Stress Model:

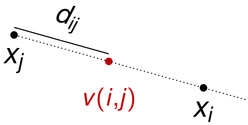
$$s'(x) = \sum_{\{i,j\} \in E} w_{ij}(||x_i - x_j|| - d_{ij})^2 + \sum_{i \in V} \sum_{p \in \mathcal{P} \setminus N(i)} w'_{ip}(||x_i - x_p|| - d_{ip})^2,$$

with adapted weights w'_{ij}

A Sparse Stress Model - Partitioning Construction

Iterative stress minimization (Gansner, Koren and North, *GD'04*, 2004):

$$x_i^\alpha = \frac{\sum_{j \neq i} w_{ij}\ v(i,j)}{\sum_{j \neq i} w_{ij}} \text{ with } v(i,j) = x_j^\alpha + \frac{d_{ij}(x_i^\alpha - x_j^\alpha)}{||x_i - x_j||}$$



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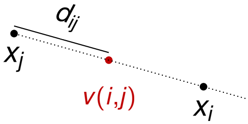
Each pivot aggregates terms $\mathcal{R}(p)$:

$$v(i,p) \approx \frac{\sum_{j \in \mathcal{R}(p)} w_{ij}\ v(i,j)}{\sum_{j \neq i} w_{ij}}$$



Requirement:

- d_{ip} similar to d_{ij} with $j \in \mathcal{R}(p)$
- x_j well distributed in close proximity around x_p

Solution:

- assign $i \in V$ to partition of closest pivot

A Sparse Stress Model - Weight Adaption

Ideal:

- $w'_{ip} = \sum_{j \in \mathcal{R}(p)} w_{ij}$

Problem:

- w_{ij}'s unknown since $w_{ij} = 1/d_{ij}^2$

Solution set $w'_{ip} = s/d_{ip}^2$:

- s number $j \in \mathcal{R}(p)$ at least as close to p as to i
  - $s = |\{j \in \mathcal{R}(p) : d_{jp} \leq d_{ip}/2\}|$

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Complexity of our Sparse Stress Model:

- preprocessing: $\mathcal{O}(k(m + n \log n))$
- time & space: $\mathcal{O}(kn + m)$

Evaluation - Setup

Data:

- 13 graphs $1K \leq n \leq 21K$ (Davis and Hu, *Sparse Matrix Collection*)
- 2 bipartite, 2 weighted graphs

Layout algorithms:

- our approach with $k \in \{50, 100, 200\}$ pivots
- maxent, GRIP, MARS, 1-stress and PivotMDS

Reported results our approach:

- median stress layout of 25 repetitions
- $\mathcal{P}$ sampled via k-means on subset of shortest-path distance matrix

Evaluation - Measures & Results

- Stress (optimally scaled)
- Time
- Procrustes Analysis
- Gabriel Graph Neighborhood preservation
- Convex Hull Classification

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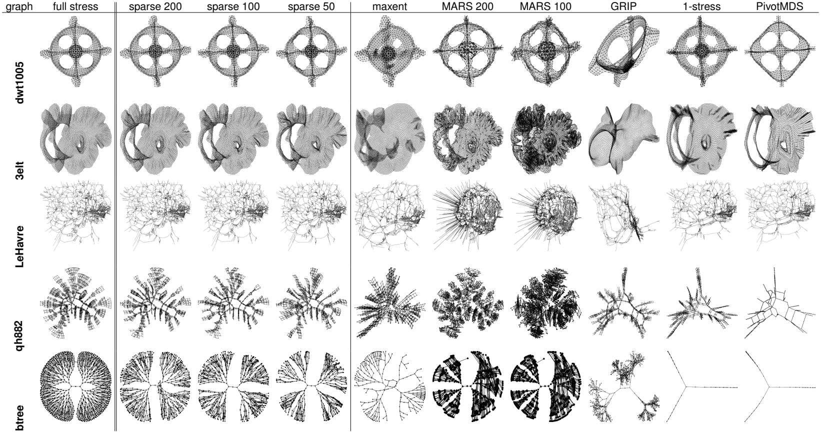
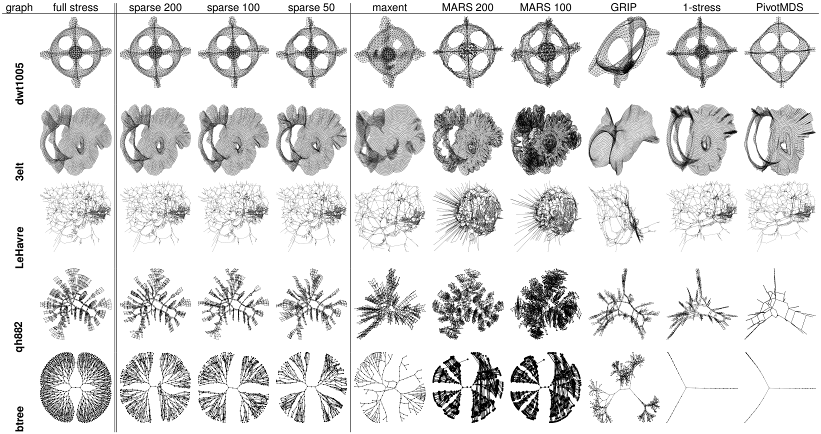
Results show that our method yields:

- lower stress
- better preservation
- higher similarity
- better classification
- lower runtime

Results


graph
full stress
sparse 200
sparse 100
sparse 50
maxent
MARS 200
MARS 100
GRIP
1-stress
PivotMDS
dwt1005
3elt
LeHavre
qh882
btree

Results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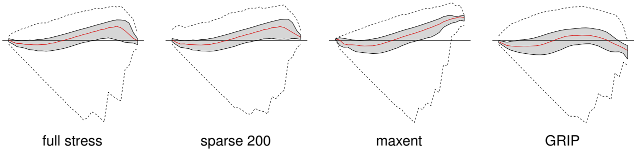
full stress
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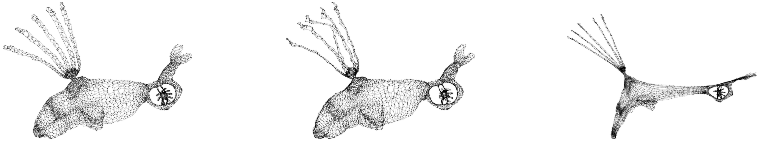
Conclusion

- 2^{nd} term makes full stress model too costly
- Sparse Stress Model: aggregation of terms
  - preprocessing: $\mathcal{O}(k(m + n \log n))$
  - time & space: $\mathcal{O}(kn + m)$
- proper pivot sampling and weight adaption allows
  - better approximation
  - less time

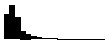
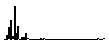
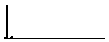


full stress (340.10s) | our approach (2.56s) | best related (3.60s)

Time Complexity

- Full Stress: $\mathcal{O}(n^2)$
- 1-stress: $\mathcal{O}(m)$
- GRIP: $\mathcal{O}(nk^2)$ with $k = \log \max\{d_{ij} : i, j \in V\}$
- maxent: $\mathcal{O}(m + n \log n)$
- MARS: $\mathcal{O}(kn + n \log n + m)$
- PivotMDS: $\mathcal{O}(nk)$
- our approach: $\mathcal{O}(nk + m)$

Evaluation - Data

graph	n	m	$\delta(G)$	$\Delta(G)$	$D(G)$	$\{deg(i)\}$	$\{d_{ij}\}$
dwt1005	1005	3808	3	26	34		
1138bus	1138	1458	1	17	31		
plat1919	1919	15240	2	18	43		
3elt	4740	13722	3	9	65		
USpowerGrid	4941	6594	1	19	46		
commanche	7920	11880**	3	3	438.00		
LeHavre	11730	15133**	1	7	33800.67		
pesa	11738	33914	2	9	208		
bodyy5	18589	55346	2	8	132		
finance256	20657	71866	1	54	55		
btree (binary tree)	1023*	1022	1	3	18		
qh882	1764*	3354	1	14	32		
lpship04l	2526*	6380	1	84	13		

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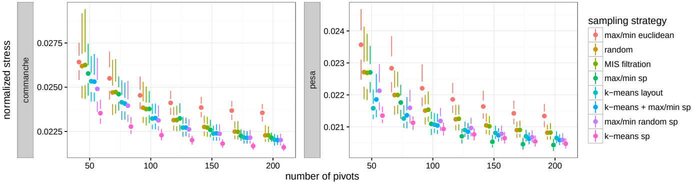
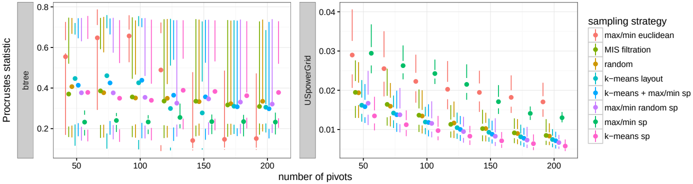
Sampling





Results

graph	full stress	our approach: sparse 200	our approach: sparse 100	our approach: sparse 50	related approaches: maxent	related approaches: MARS 200	related approaches: MARS 100	related approaches: GRIP	related approaches: 1-stress	related approaches: PivotMDS
stress										
dwt1005	10 729	**10 940**	**11 081**	**11 329**	21 623	17 660	20 134	52 517	12 495	14 459
3elt	422 940	**426 564**	**430 200**	**437 051**	585 967	503 600	754 134	934 206	555 934	634 401
commanche	654 694	**677 220**	**699 890**	**749 609**	1 507 654	2 761 605	3 145 489	1 539 767	2 085 818	2 157 943
LeHavre	439 188	**433 030**	**441 986**	**454 785**	1 231 283	12 012 307	12 570 692	8 658 371	1 255 474	1 305 577
pesa	1 373 514	**1 417 449**	**1 452 975**	**1 495 512**	10 423 779	3 563 772	8 281 116	2 957 738	3 486 176	3 325 889
finance256	6 175 210	**6 415 761**	**6 474 787**	**6 582 890**	8 151 335	7 267 598	8 643 239	19 817 355	12 257 268	11 380 089
btree	60 206	**61 839**	**63 325**	**66 122**	67 871	103 436	100 767	96 235	157 988	164 329
Procrustes statistic										
dwt1005		**0.001**	**0.005**	**0.003**	0.027	0.008	0.018	0.263	0.004	0.008
3elt		**0.001**	**0.001**	**0.002**	0.026	0.009	0.029	0.199	0.017	0.023
commanche		**0.001**	**0.002**	**0.005**	0.039	0.026	0.167	0.092	0.066	0.066
LeHavre		**0.001**	**0.001**	**0.001**	0.012	0.163	0.173	0.256	0.010	0.010
pesa		**0.009**	**0.010**	**0.010**	0.095	0.025	0.070	0.017	0.021	0.021
finance256		0.009	**0.006**	**0.005**	0.013	0.007	0.018	0.206	0.042	0.041
btree		0.748	**0.165**	0.241	0.233	0.360	0.367	0.386	0.361	0.364
time in seconds										
dwt1005	1.26	0.22	0.14	**0.10**	0.47	1.02	2.36	0.06	0.08	0.06
3elt	31.82	0.94	0.59	**0.46**	2.26	16.31	8.43	0.71	0.37	0.23
commanche	340.10	4.89	2.56	**1.47**	3.60	22.72	12.43	2.29	0.47	0.35
LeHavre	475.05	6.53	3.48	**2.37**	6.31	27.57	19.50	10.18	0.81	0.54
pesa	373.23	4.25	2.47	**1.53**	5.96	50.10	42.68	3.56	0.95	0.60
finance256	1016.92	7.32	4.41	**3.09**	14.76	32.16	24.66	12.12	2.51	1.60
btree	7.79	0.38	0.15	**0.10**	0.63	2.70	1.48	0.06	0.06	0.03

Summary:

- lower stress
- higher similarity
- less time

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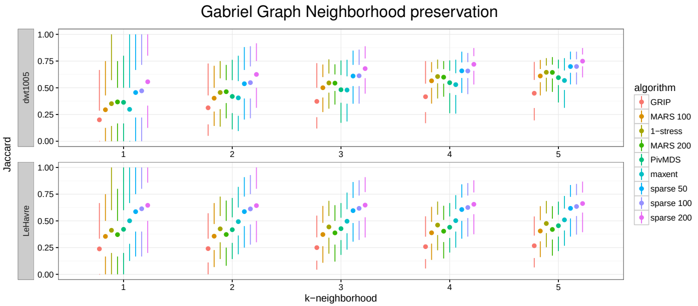
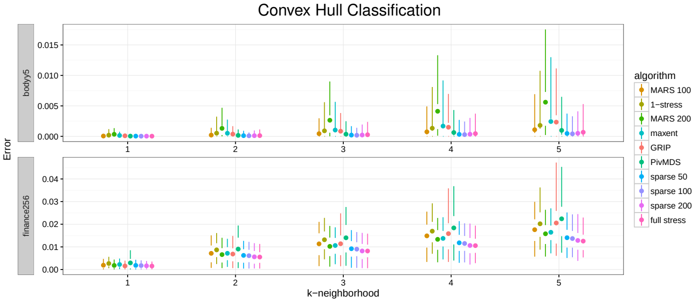
Results





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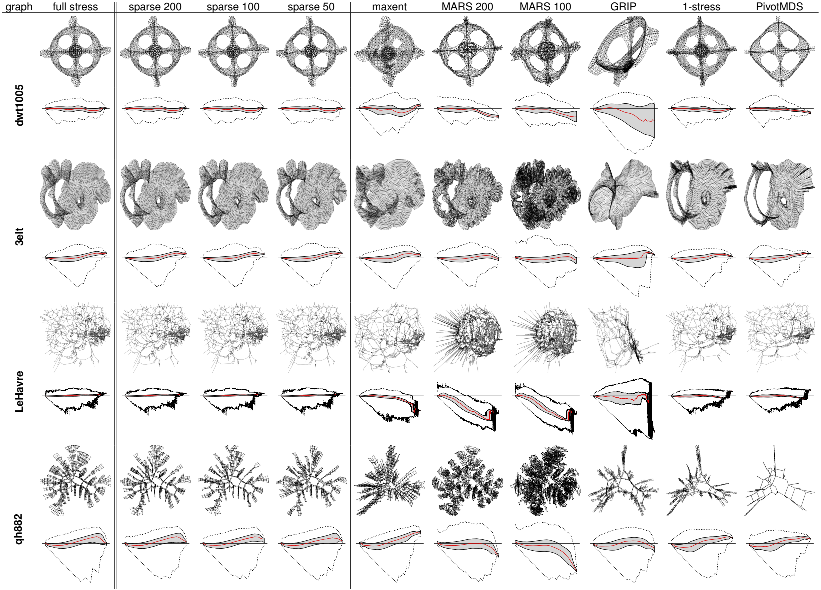
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