Twins in Subdivision Drawings of Hypergraphs

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24th International Symposium on
Graph Drawing & Network Visualization
19-21 September 2016

Subdivision Drawings of Hypergraphs

Hypergraph: Vertices and hyperedges (vertex subsets)

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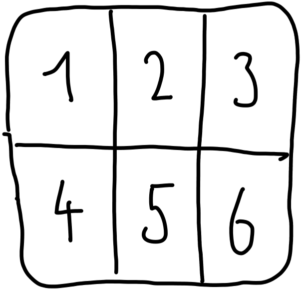

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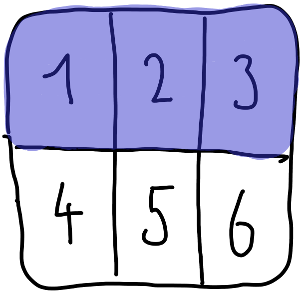

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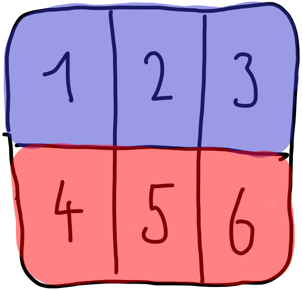

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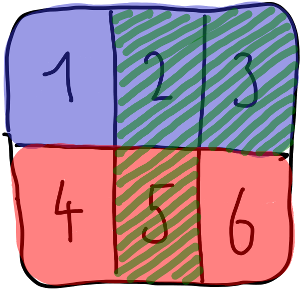

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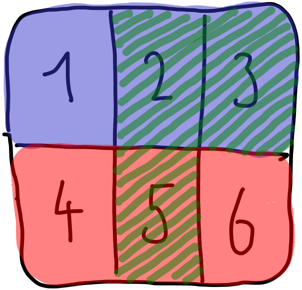

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[Johnson and Pollak, 1987]
[Mäkinen, 1990]
[Kaufmann et al. 2008]
[Brandes et al. 2010]
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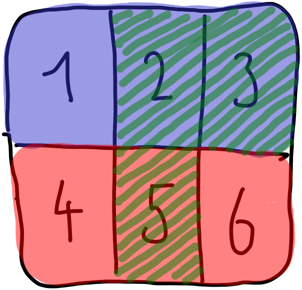

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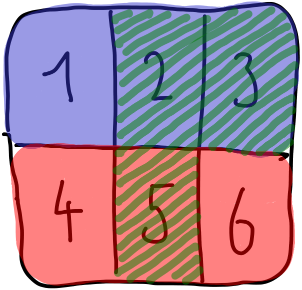

- Intuitive approach: Draw twins adjacently

⇝ Remove twins initially, add them back in later
[Mäkinen 1990, Kaufmann et al. 2008, Buchin et al. 2011]

New results

- Give hypergraph $\mathcal{H}$ with two twins t, t' such that
  - $\mathcal{H}$ has a subdivision drawing,
  - $\mathcal{H} - t$ and $\mathcal{H} - t'$ do not have subdivision drawings.

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  More than $f(m)$ twins in $\mathcal{H} \Rightarrow$
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- r layers in the drawing, m hyperedges $\Rightarrow$
  $f(m, r) = 2^{2^{O(mr^2 \log r)}}$.
  - $\rightsquigarrow$ Obtain such r-layered drawings efficiently for small r and number m of hyperedges

Planar Supports

A **support** for a hypergraph is a graph G on the same vertex set such that each hyperedge induces a connected subgraph of G.

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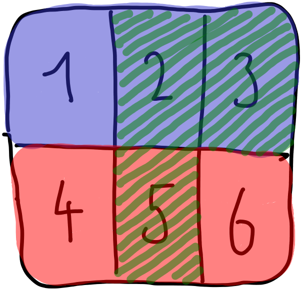

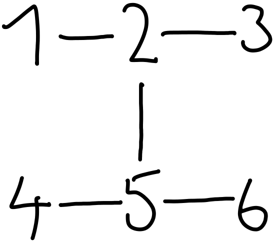

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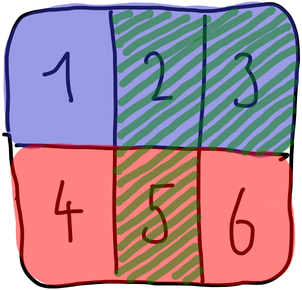
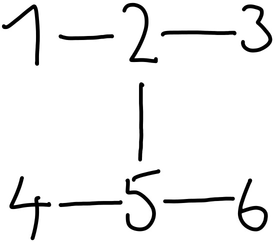
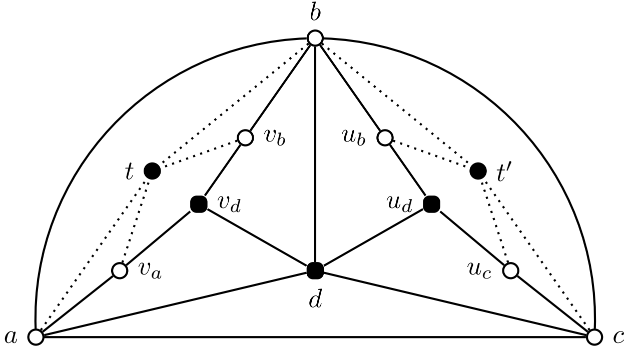




Theorem (Johnson and Pollak, 1987)

A hypergraph has a subdivision drawing if and only if it has a planar support.

A hypergraph with a planar support and two twins t, t'



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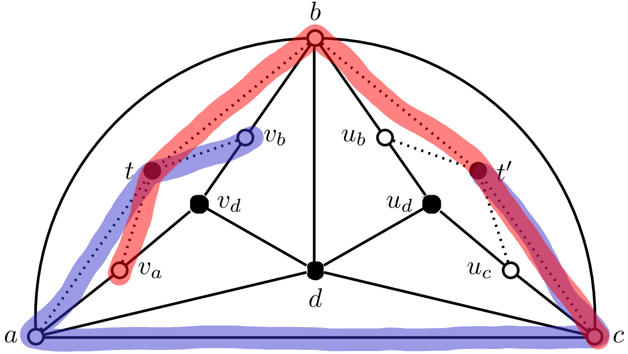

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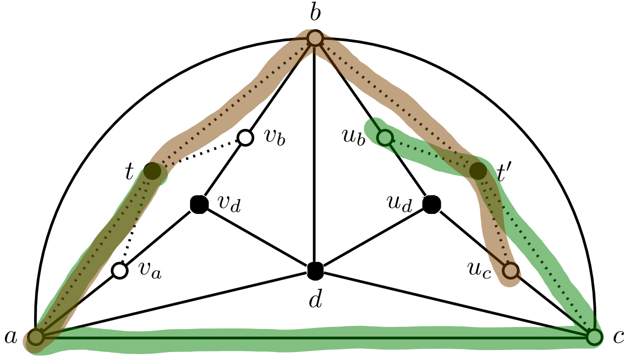

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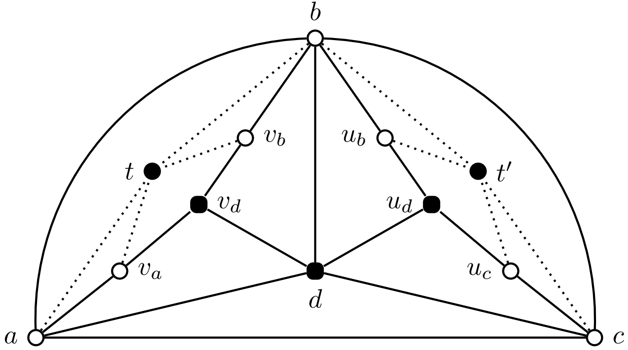

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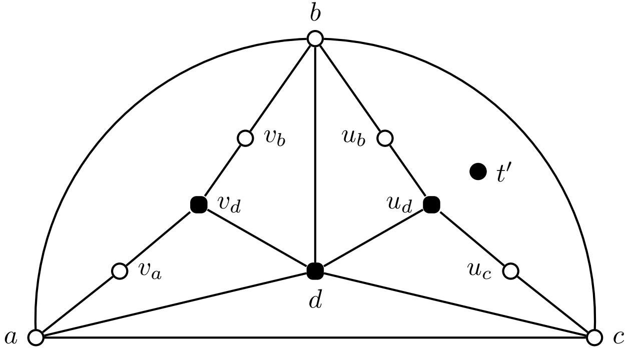

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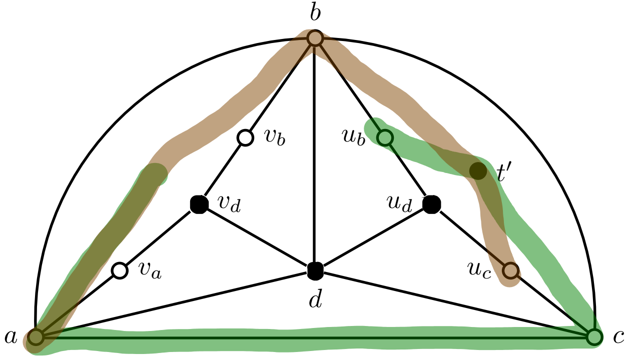

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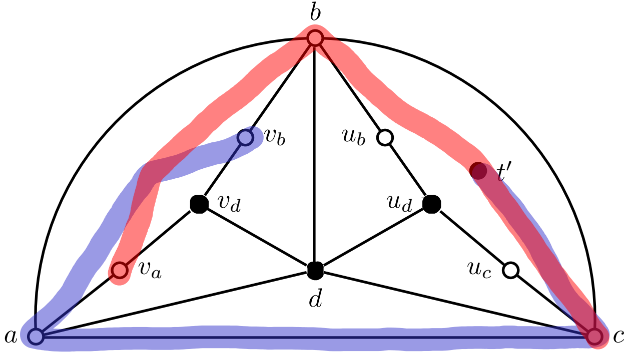

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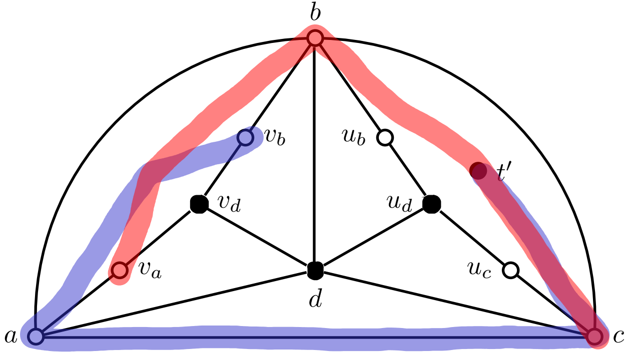

Beware! (Nonadjacent) Twins may be necessary for a planar support!

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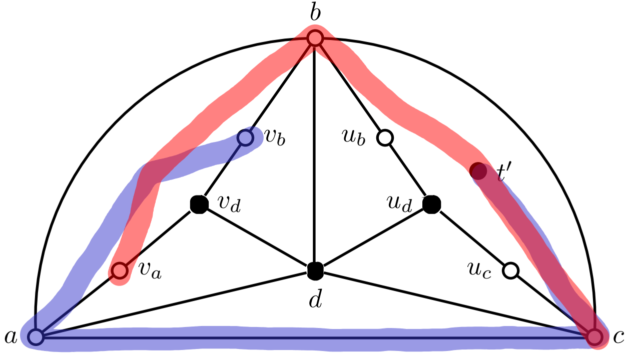

Beware! (Nonadjacent) Twins may be necessary for a planar support!
Generalizes to arbitrarily large twin class.

The number of important twins

Are **all** twins important though?

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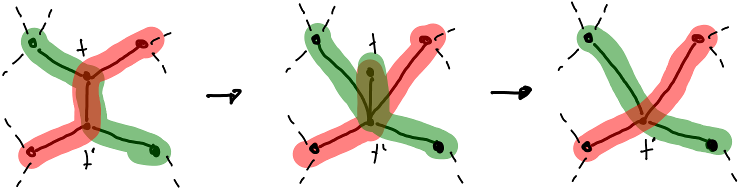
Aim to find drawing with ***r*-layers**/an ***r*-outerplanar** support for hypergraph with m hyperedges:

Reduction rule

If there is a twin class containing $2^{r^{\Omega(r^2 m)}}$ twins,
then remove one of them.

Finding an r-outerplanar support for a sparse hypergraph



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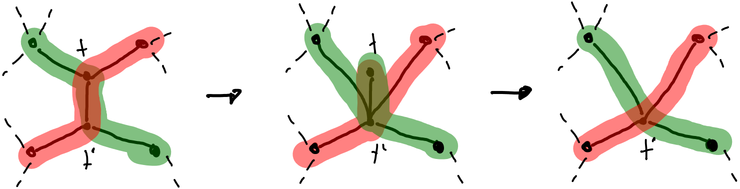
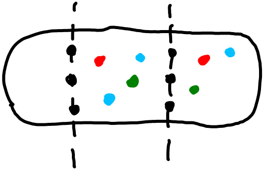


Prove:

Large class of mutual twins
⇝ *r*-outerplanar support with two adjacent twins.

Rewrite an r-outerplanar support

Suppose we have two separators as follows:



1. Each middle vertex has a twin on the right
2. After removing everything between the separators and gluing the remaining parts on the separators:
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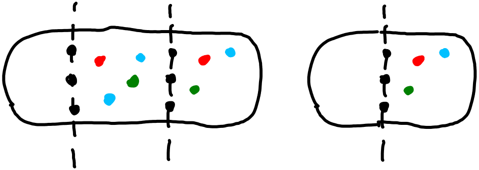

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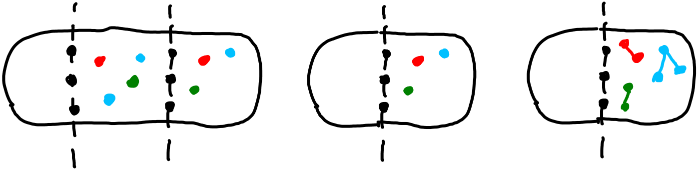

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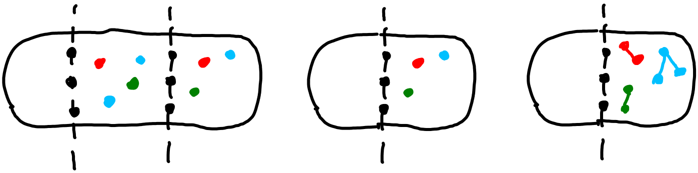

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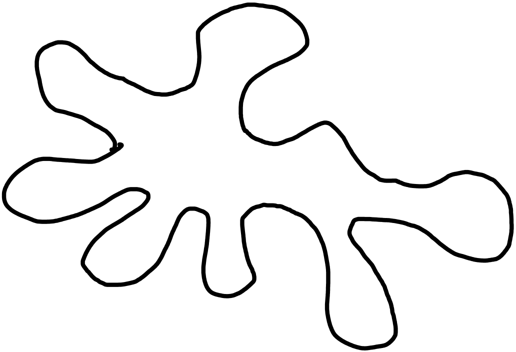
Show: Large class of mutual twins $\rightsquigarrow$ two separators as above.

r-Outerplanar graphs are tree-like



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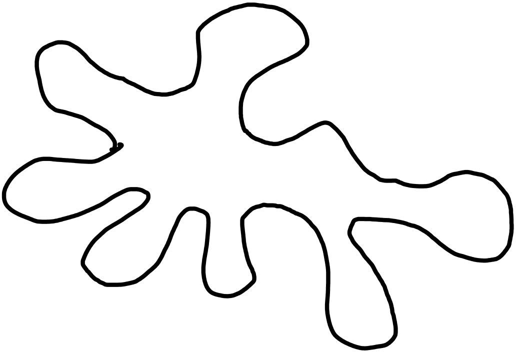

Theorem

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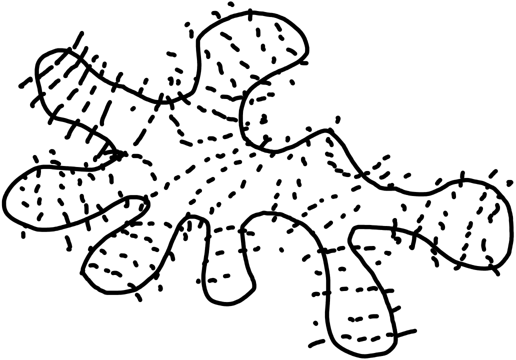

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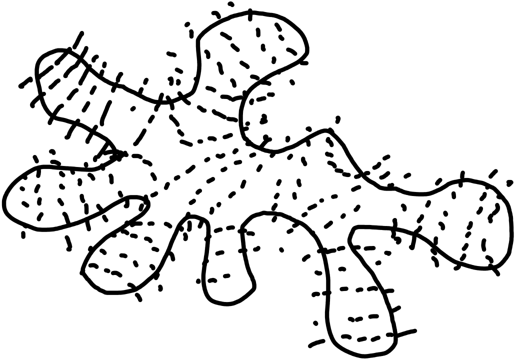

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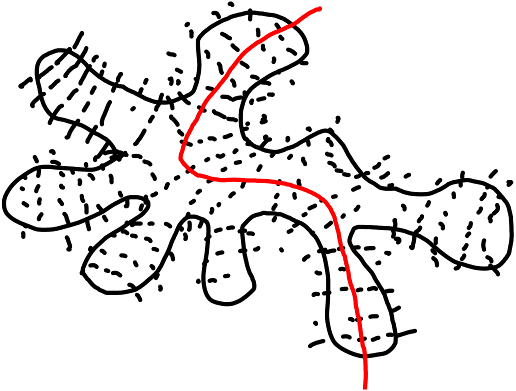

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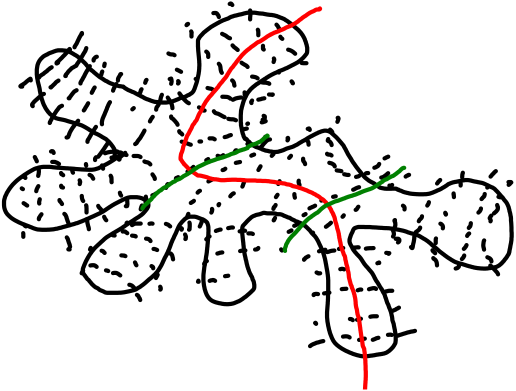

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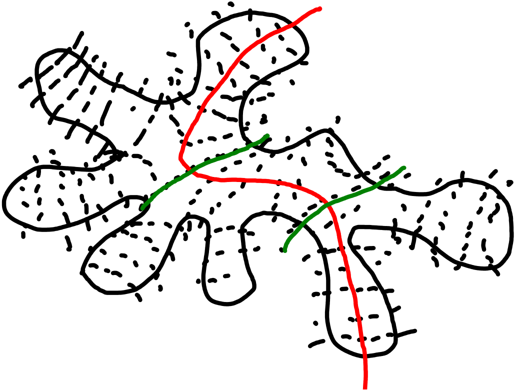

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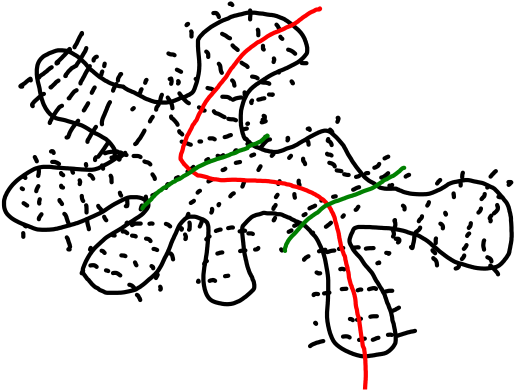

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m = number of hyperedges

Conclusion and Outlook

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