C-planarity of Embedded Cyclic c-Graphs

Radoslav Fulek,IST Austria

C-planarity & Approximating Maps

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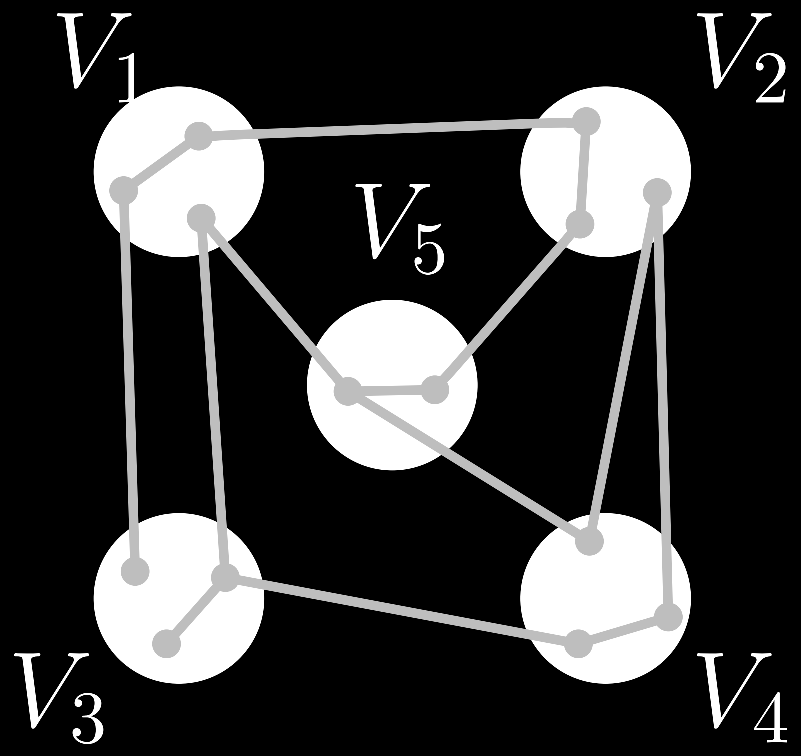

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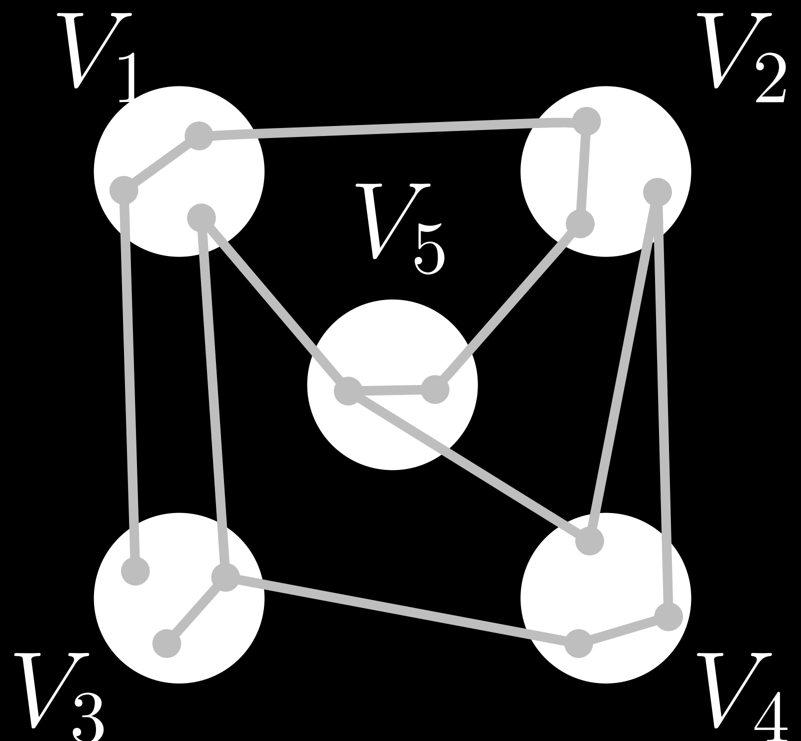

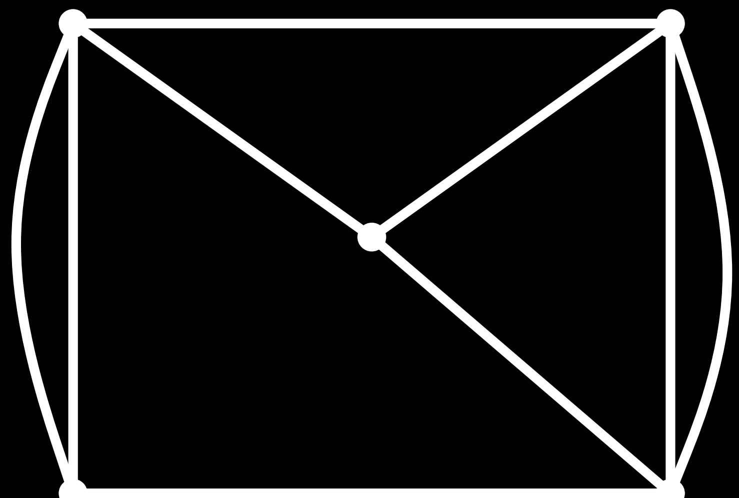

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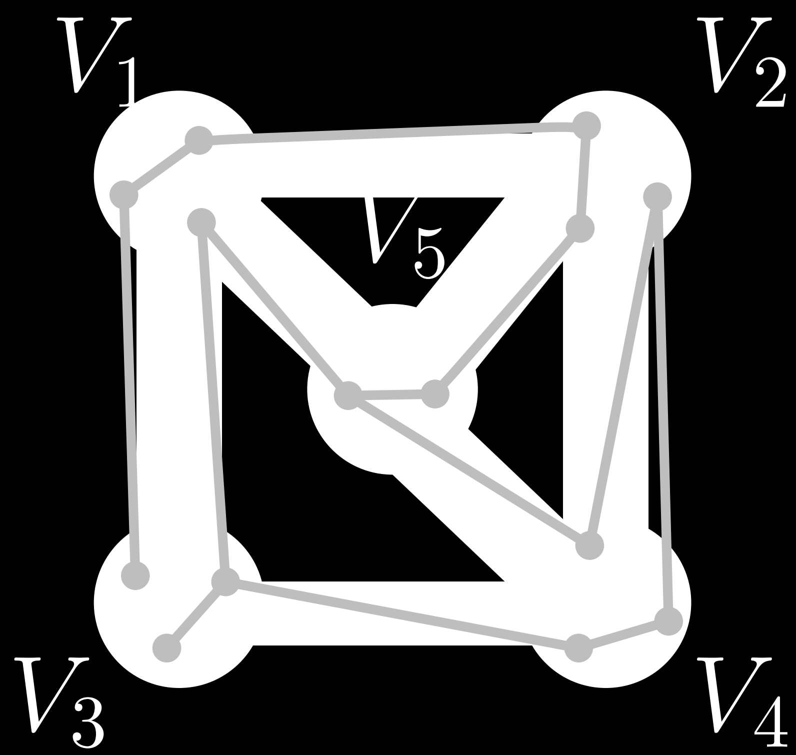

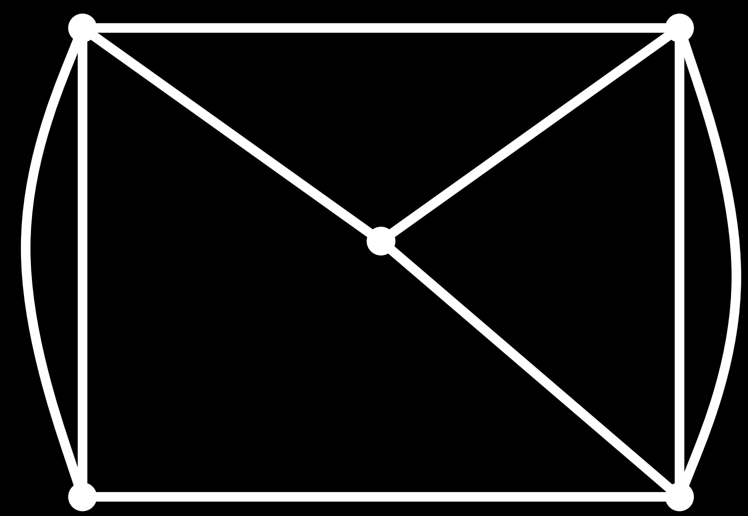

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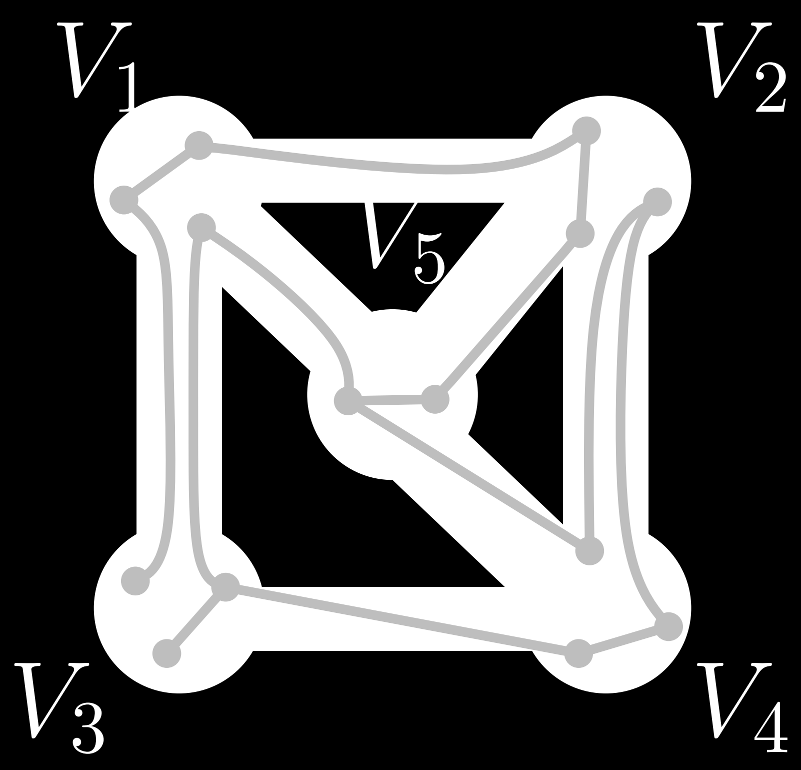

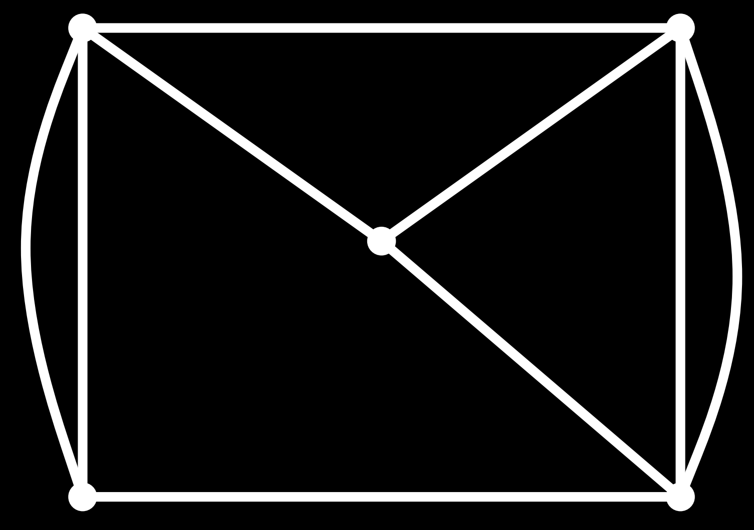

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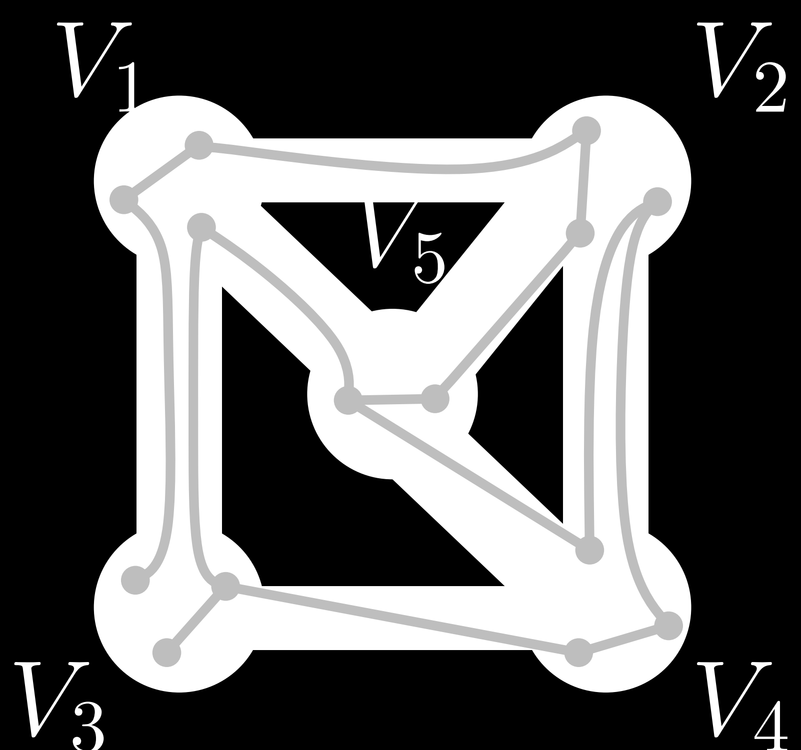

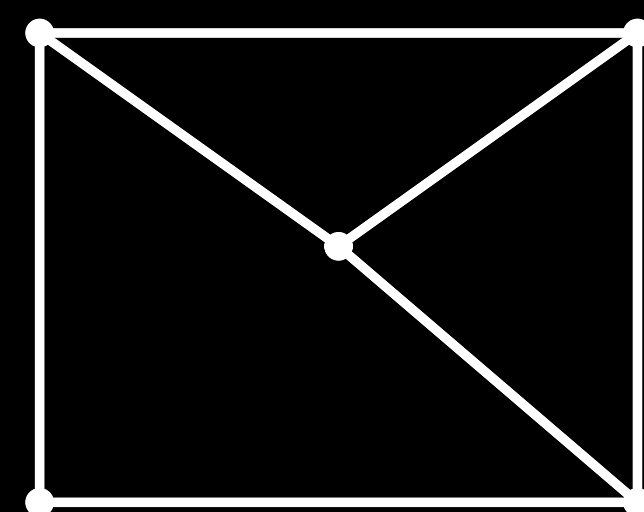

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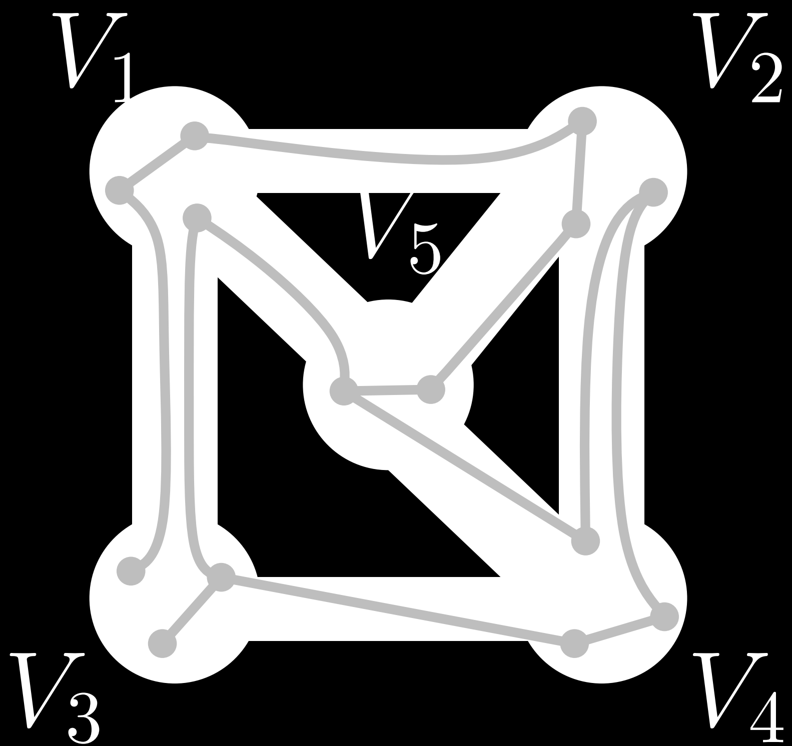

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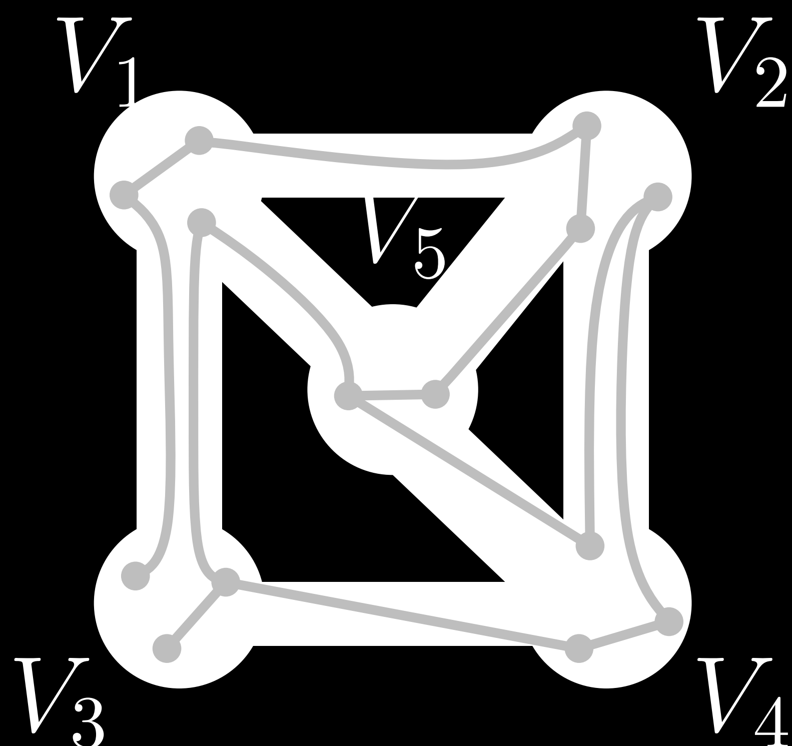

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C-planarity with pipes is tractable for cycles.

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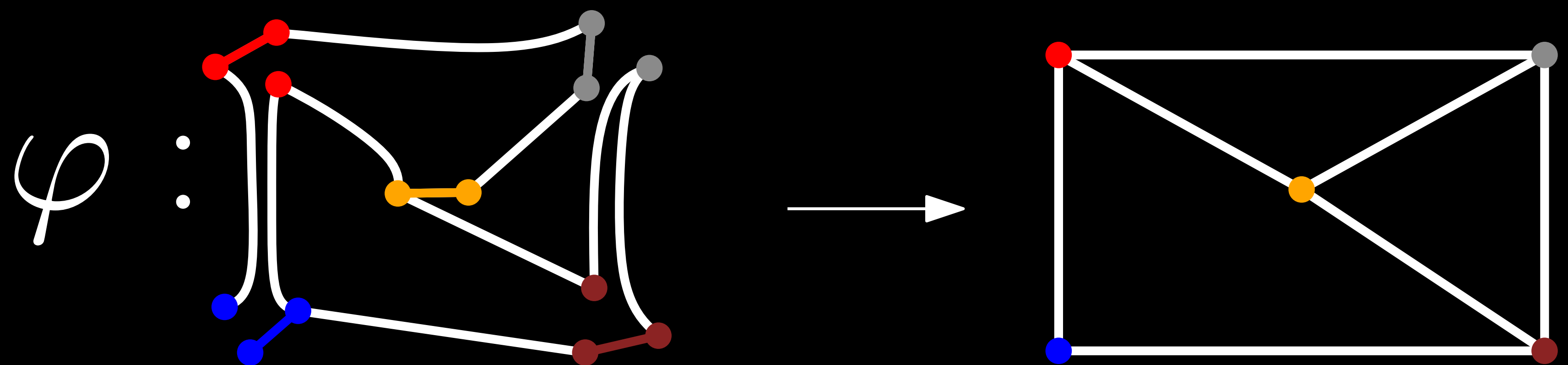

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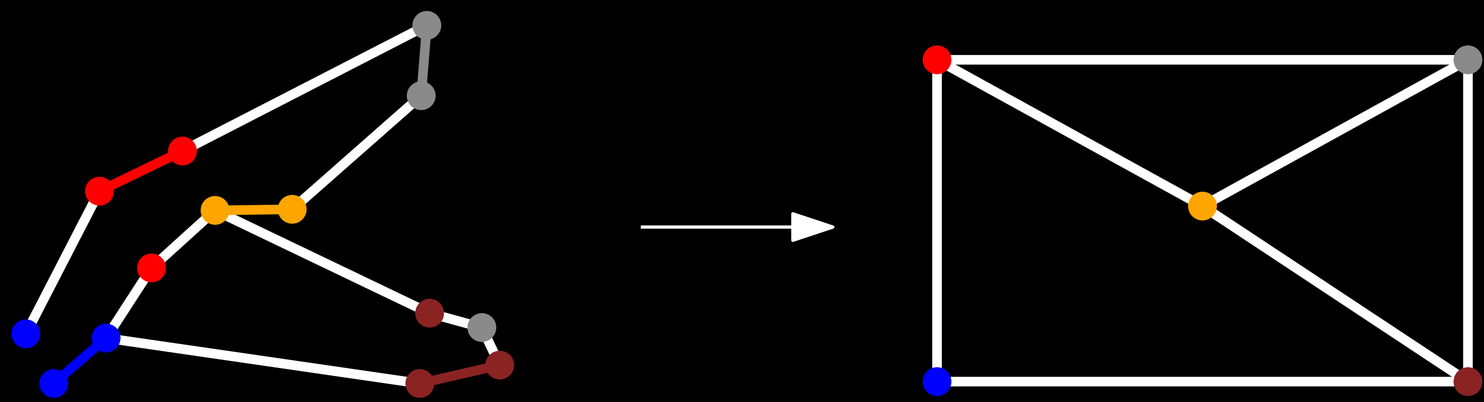

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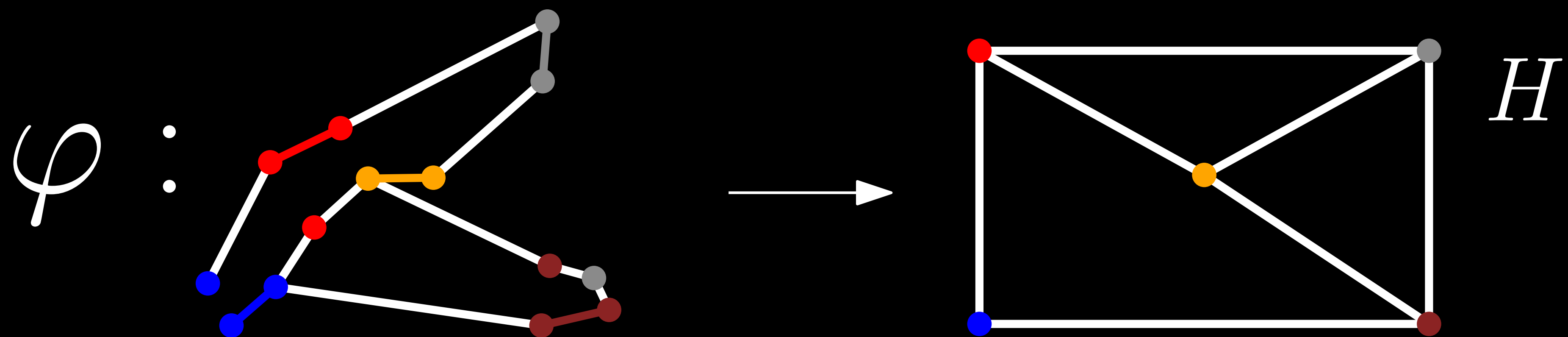

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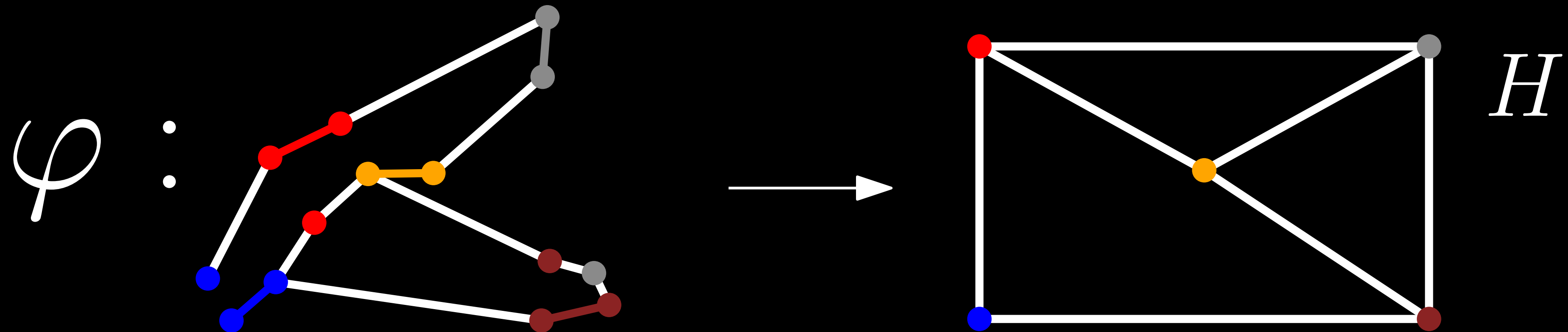

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φ **is approximable by an embedding** if there exists an ϵ-approximation that is an embedding for every $\epsilon > 0$.

Warm up

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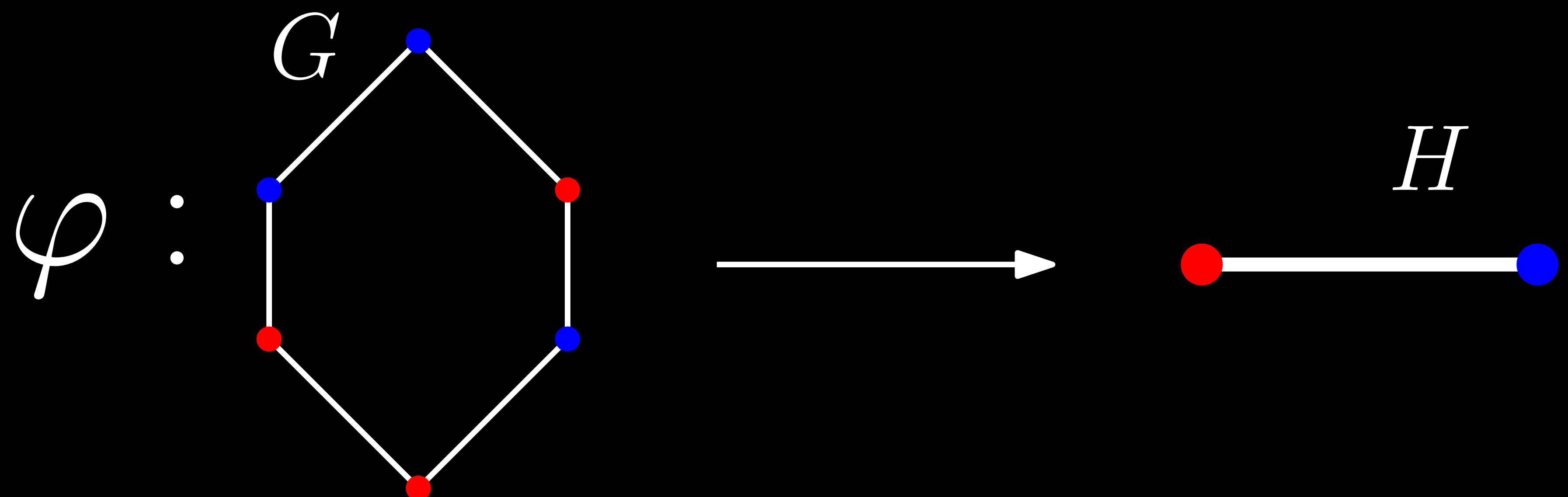

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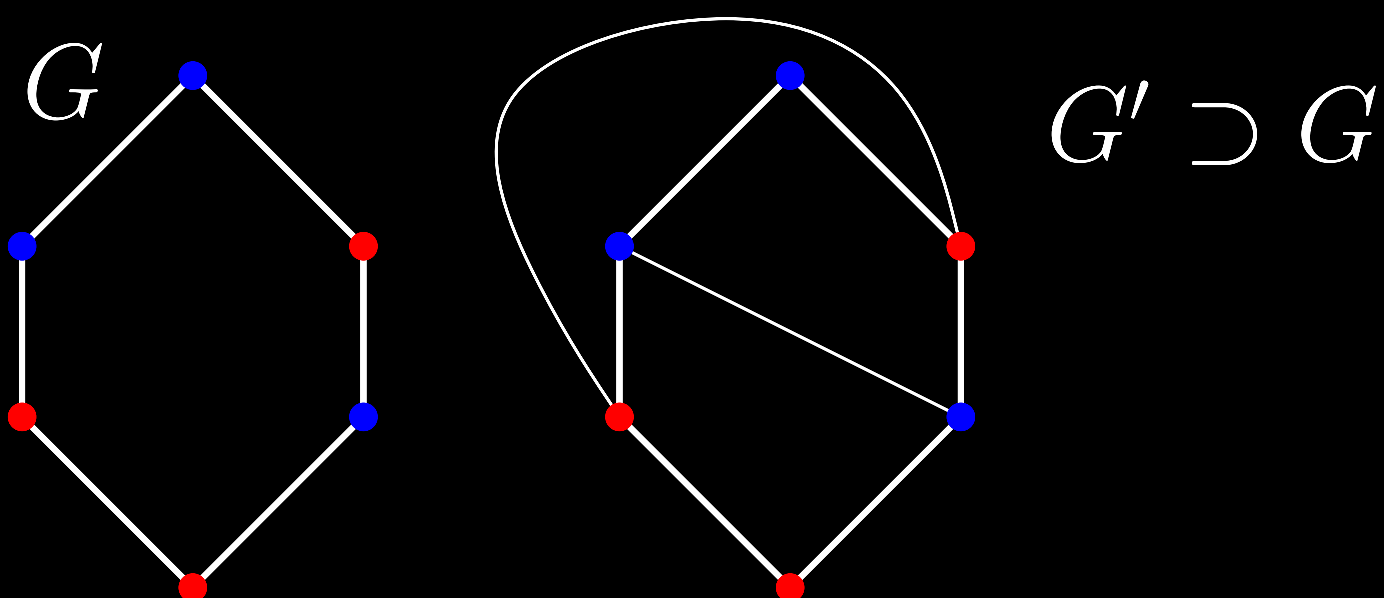

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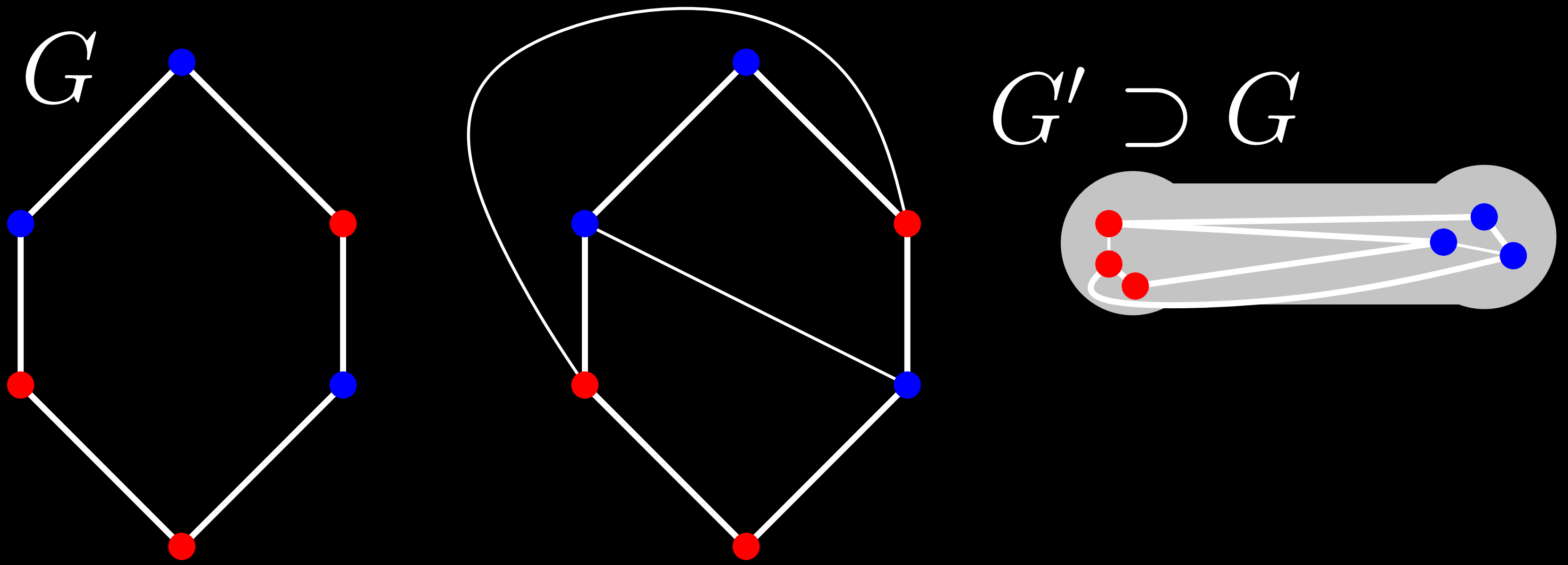

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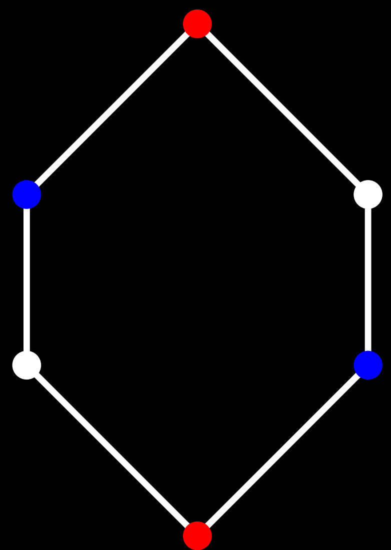

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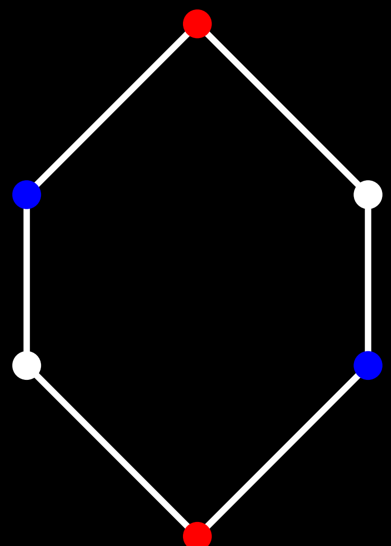

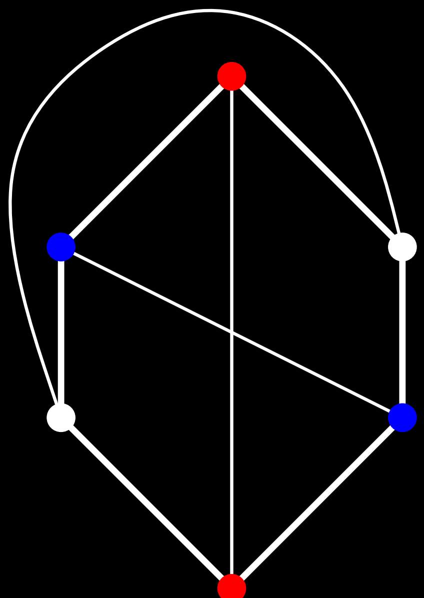

$K_{3,3}$

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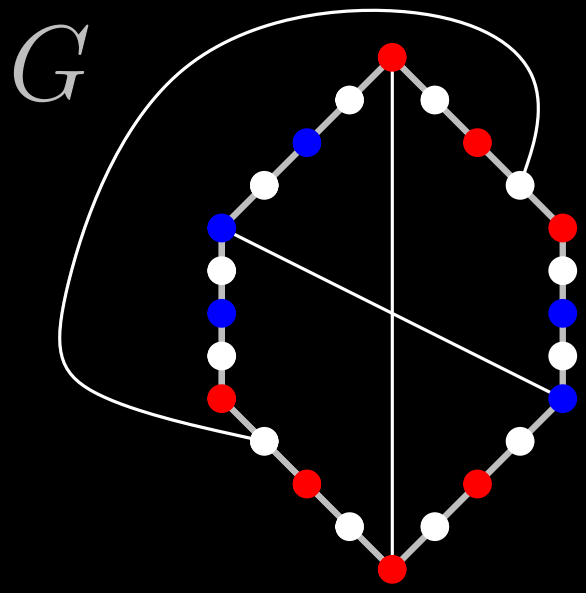

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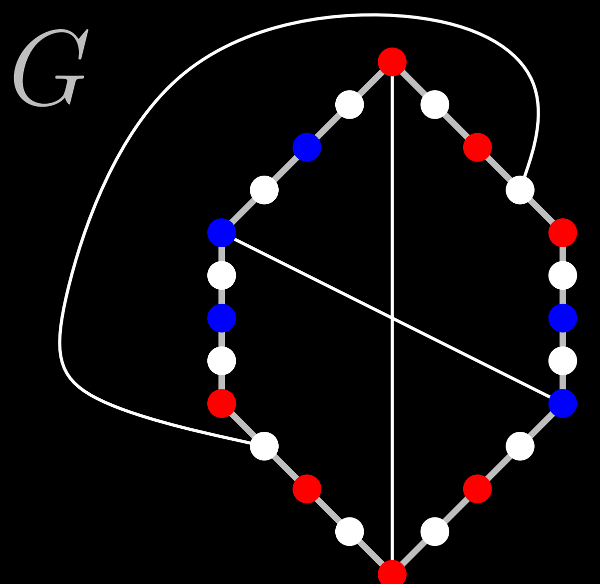

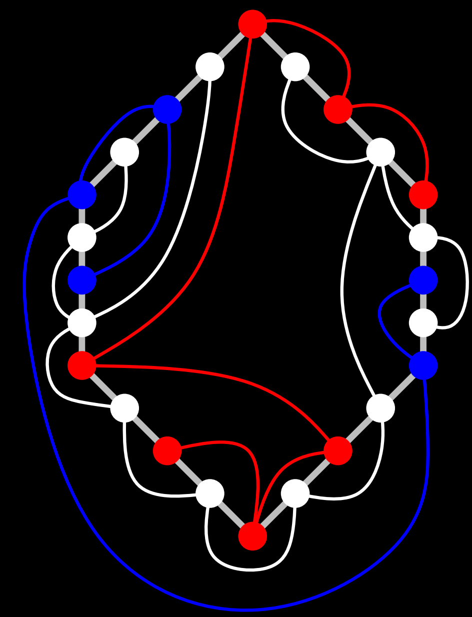

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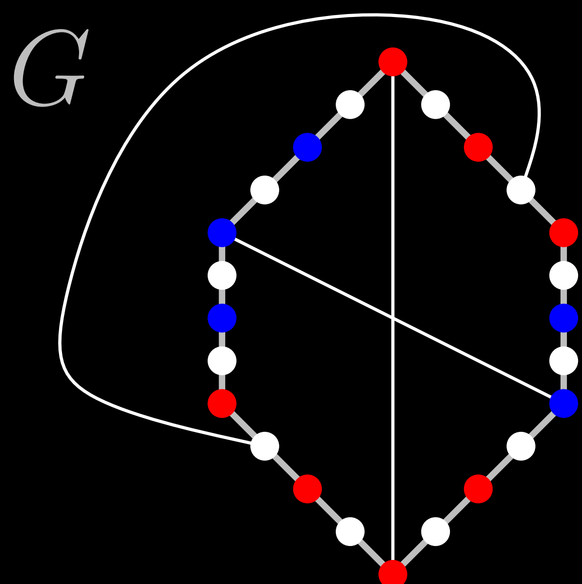

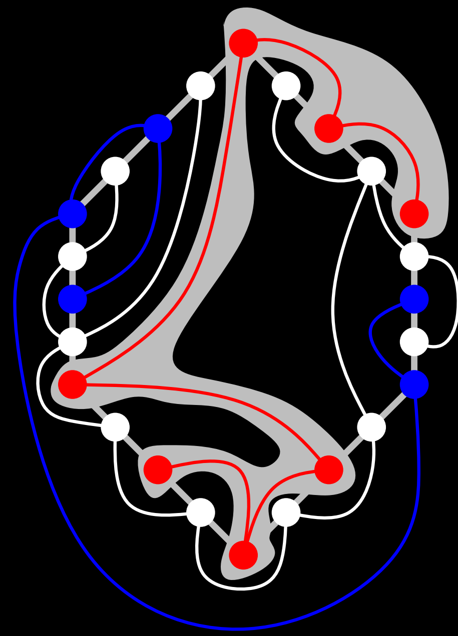

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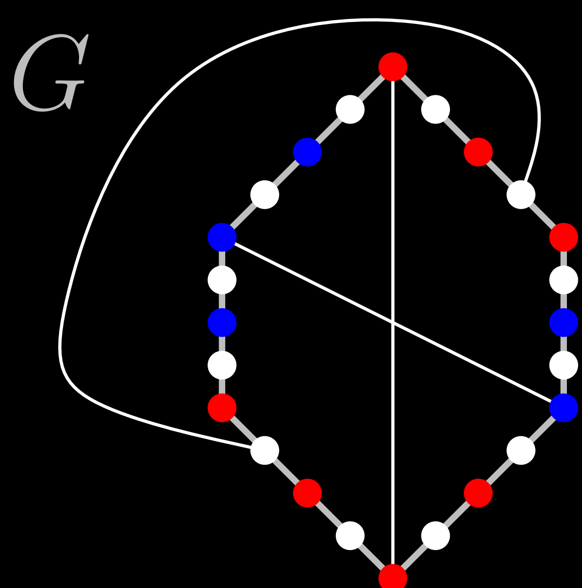

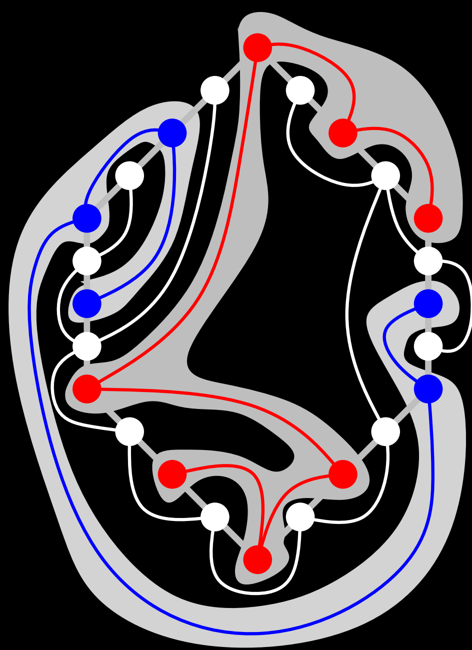

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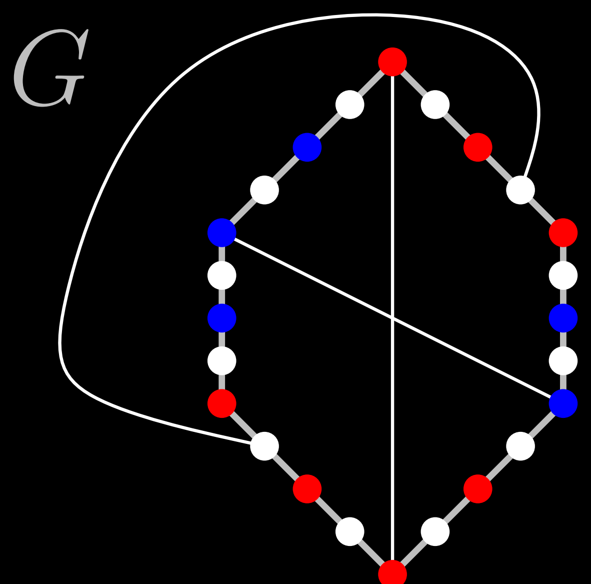

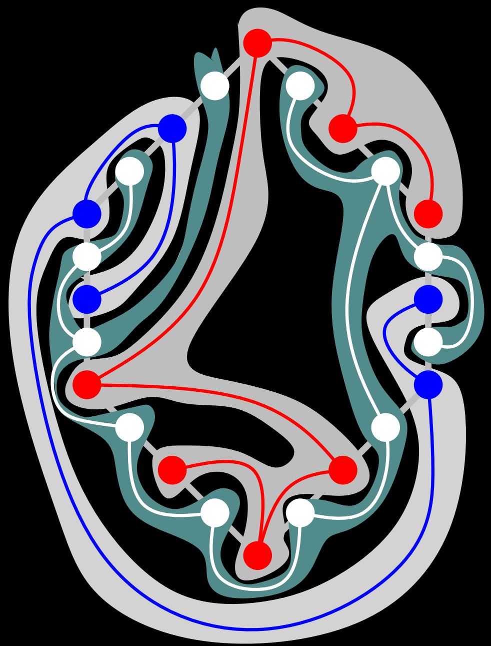

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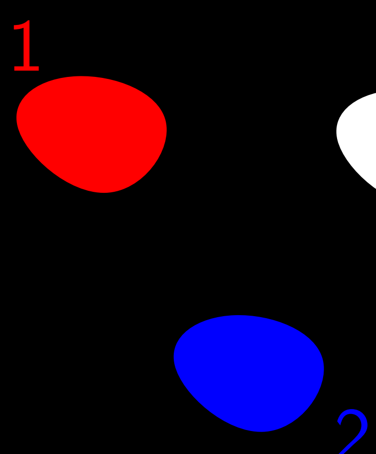

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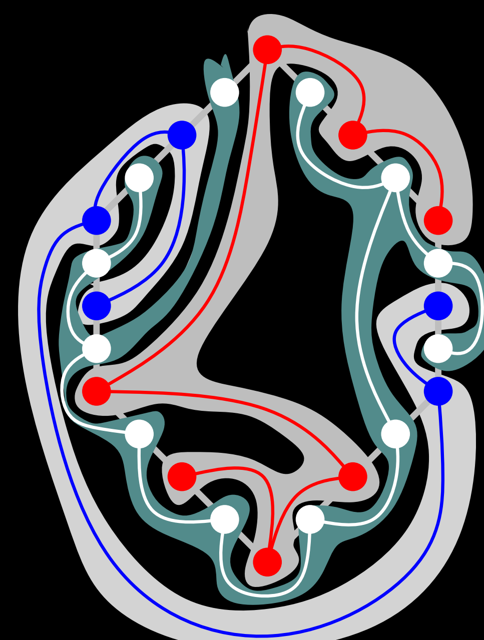

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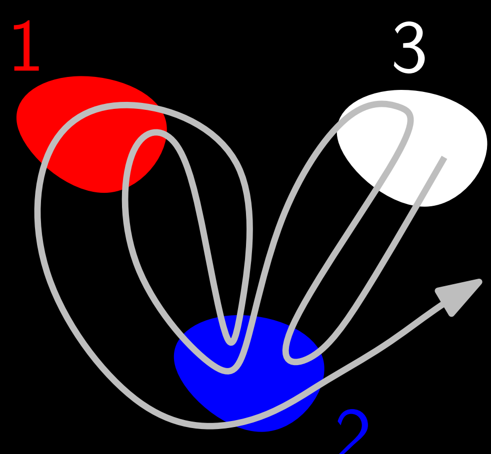

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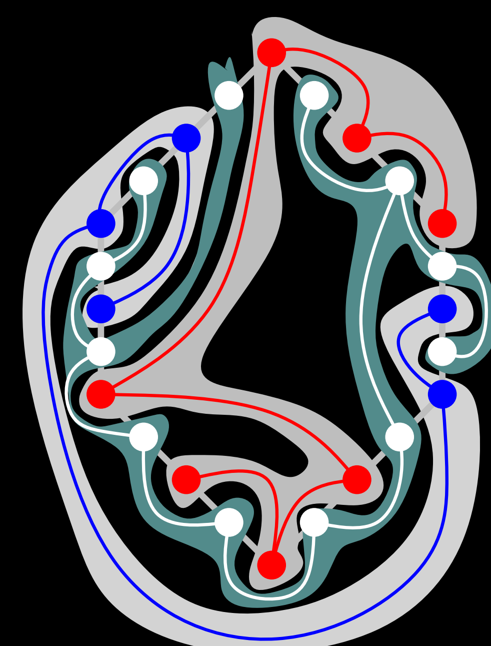

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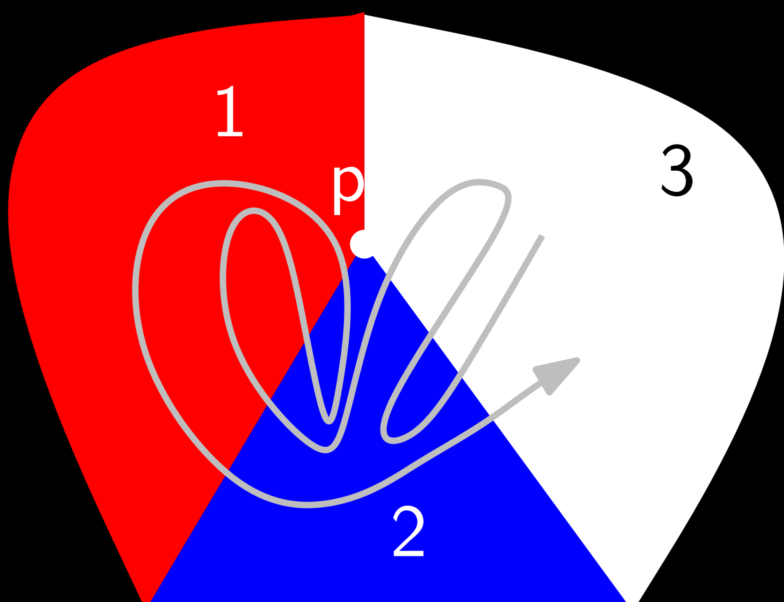

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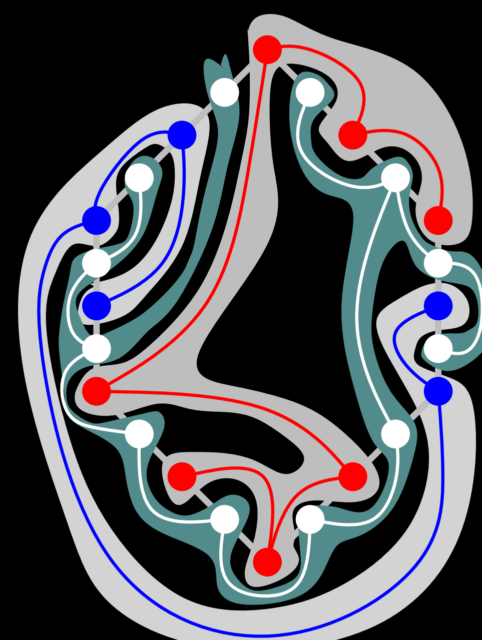

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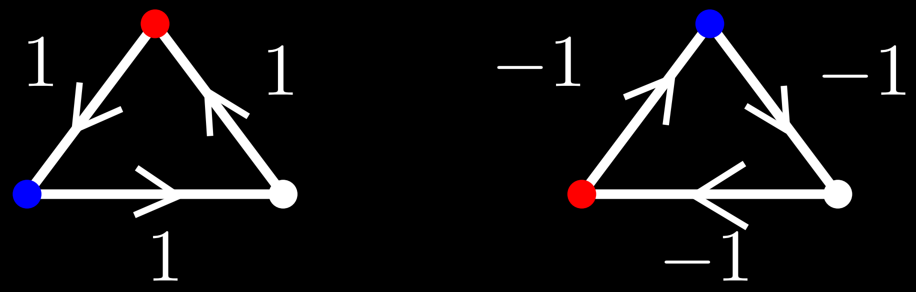

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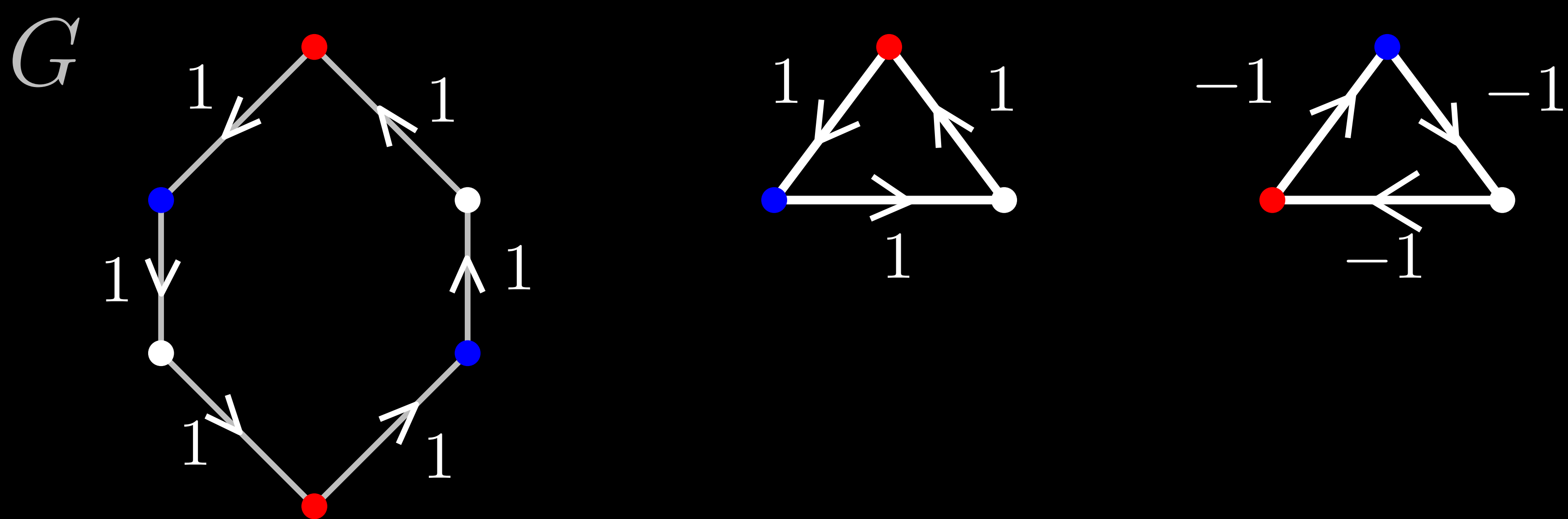

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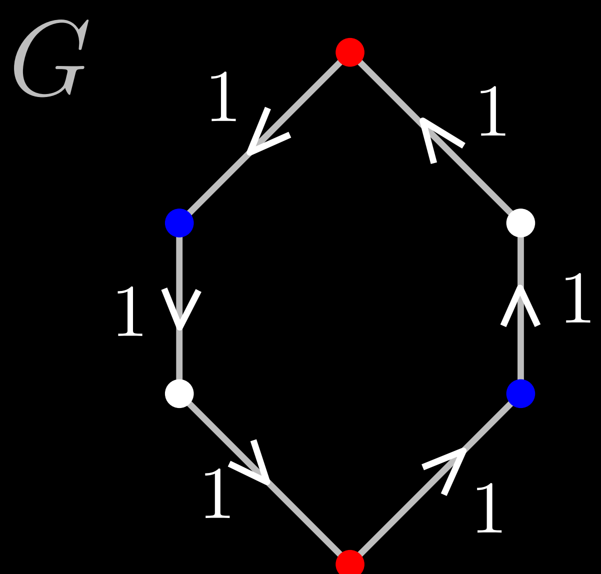

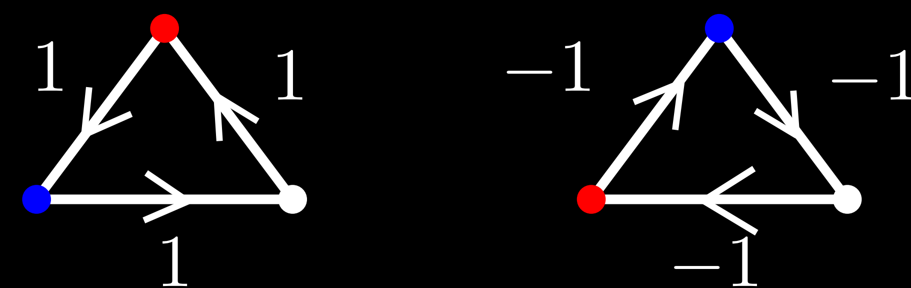

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Sum labels and devide by three, $wn = \frac{\sum_e l(e)}{3}$. The instance is positive iff $|wn| \leq 1$.

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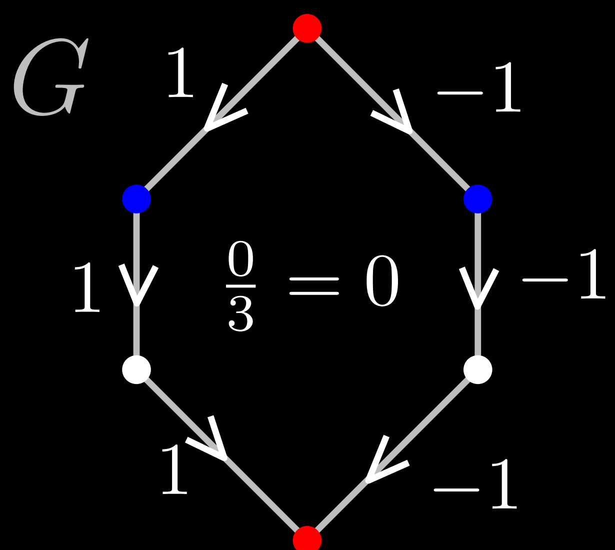

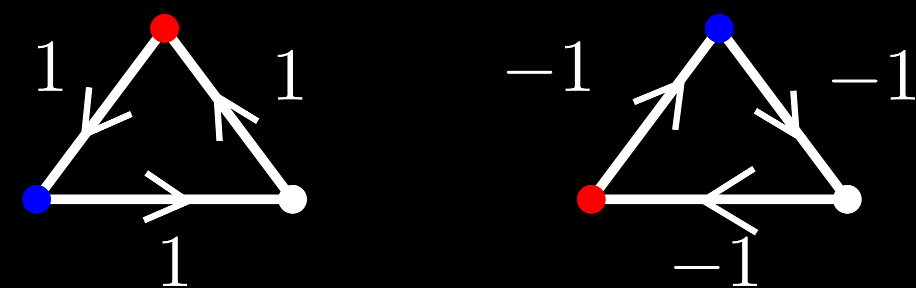

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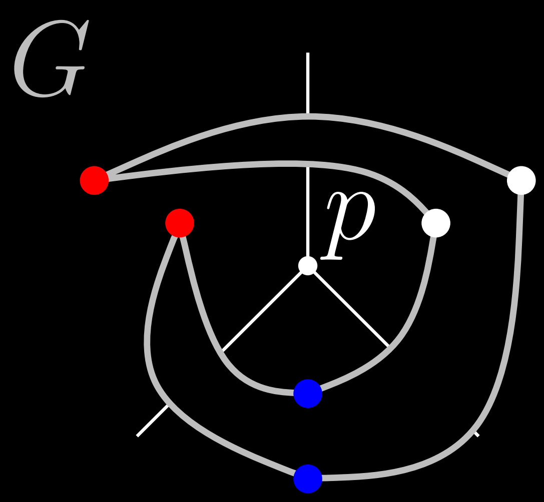

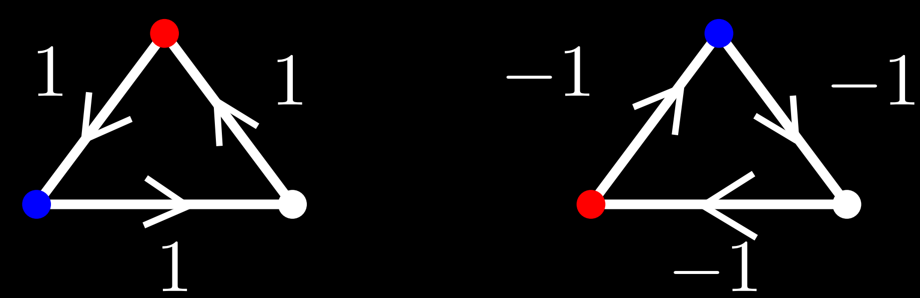

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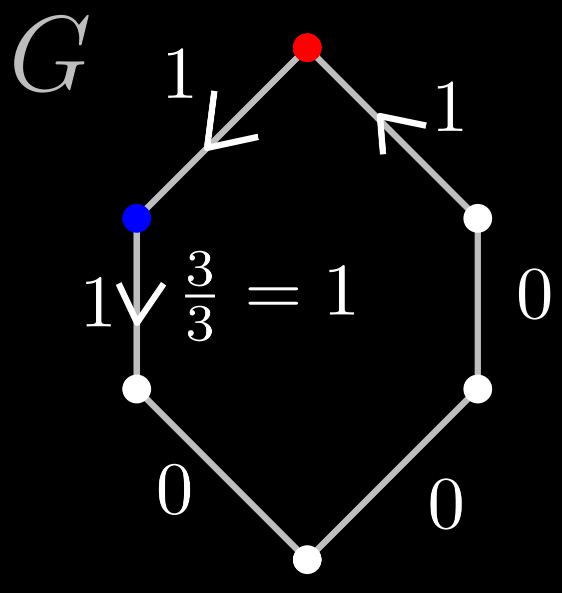

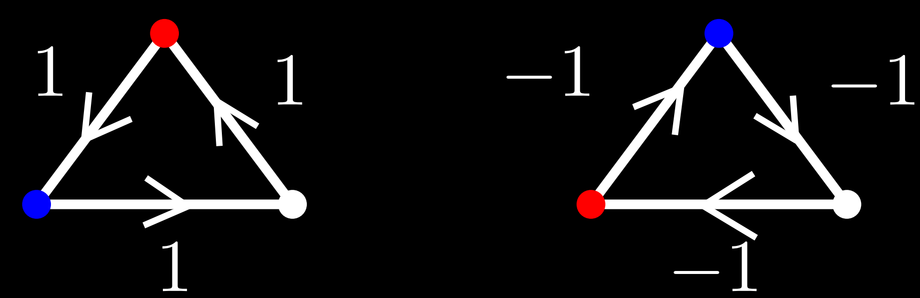

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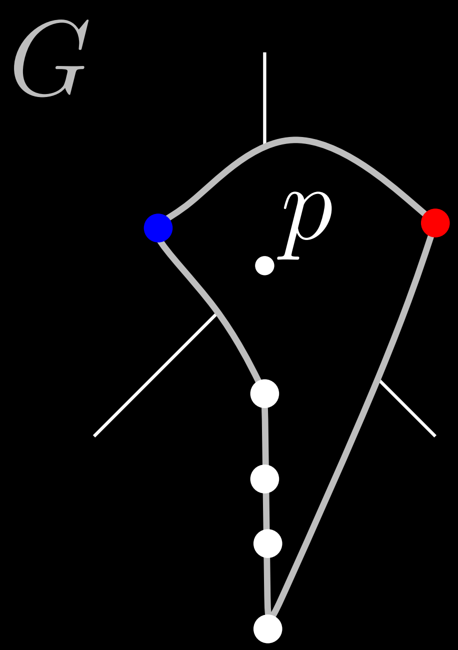

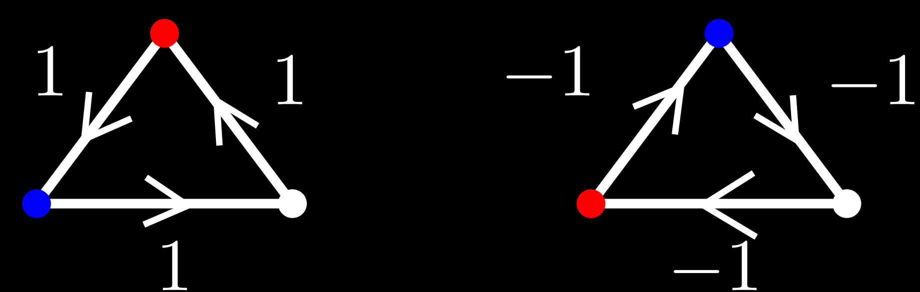

Algorithm: (Cortese et al. 2005)
Sum labels and devide by three, $wn = \frac{\sum_e l(e)}{3}$. The instance is positive iff $|wn| \leq 1$.

C-planarity of Emedded Cyclic c-Graphs

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Let $G = (V_1 \uplus V_2, E)$ be a planar graph mapped by φ to an edge.

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Theorem 3. Biedl (1998), Gutwenger et al. (2002) *We can decide in polynomial time if φ is approximable by an embedding.*

C-planarity of Emedded Cyclic c-Graphs

Let $G = (V_1 \uplus V_2 \uplus \ldots \uplus V_k, E)$ be a planar graph mapped by φ to a k-cycle.

Theorem 1. **(F.)** *We can decide in quadratic time if φ is approximable by an embedding with a given rotation system.*

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Sieklucki (1969) characterized all the graphs G for which any map $\varphi : G \to \mathbb{R}$ is approximable in $\mathbb{R}^2$.

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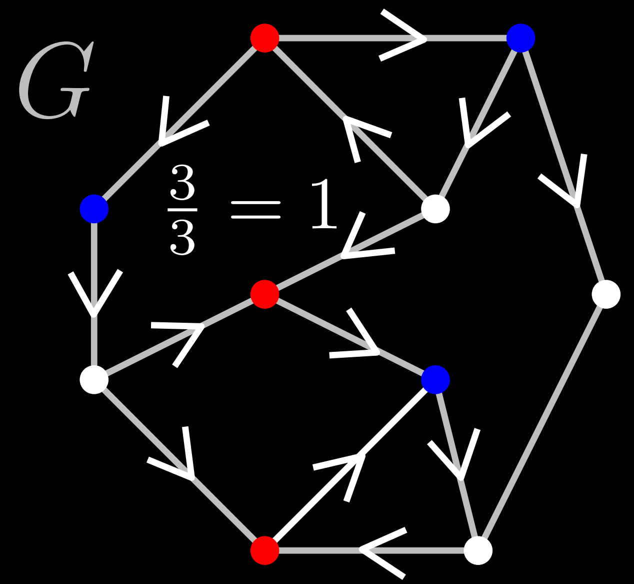

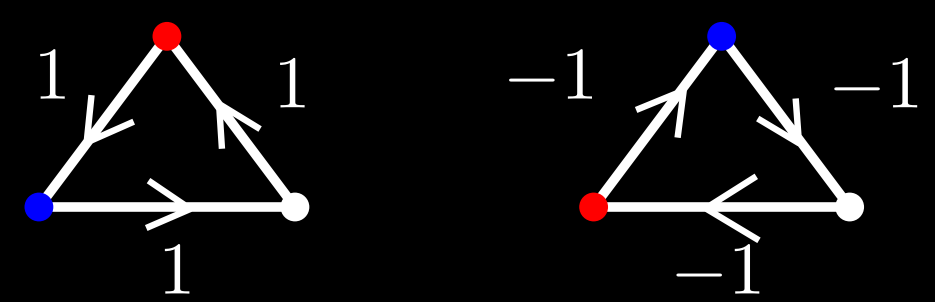

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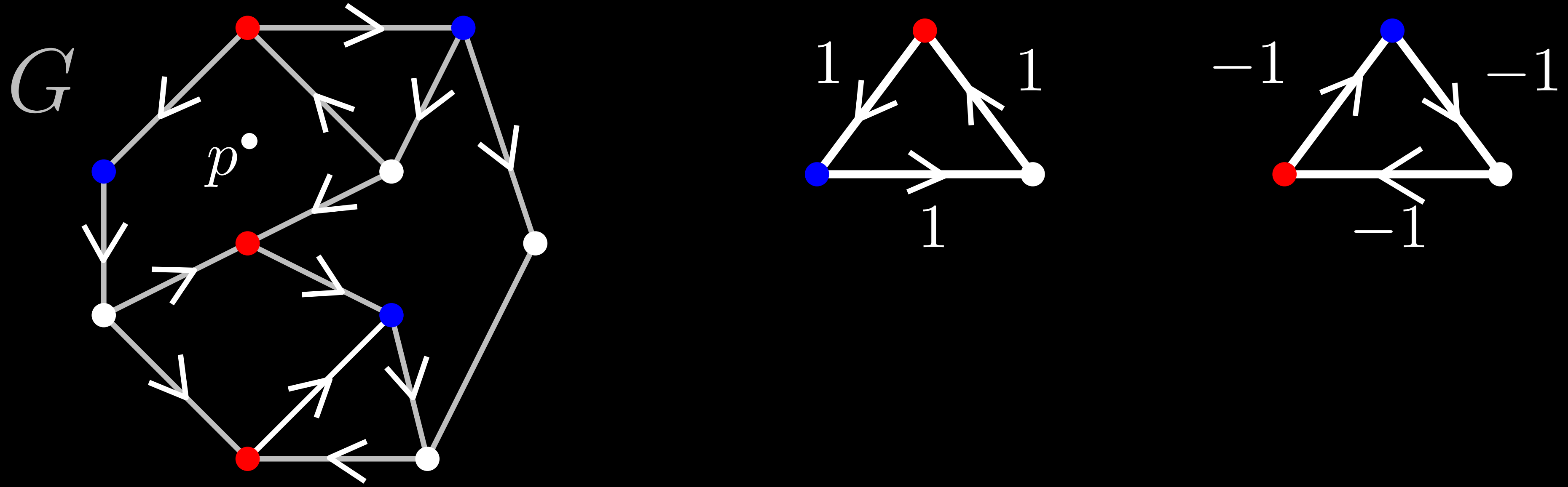

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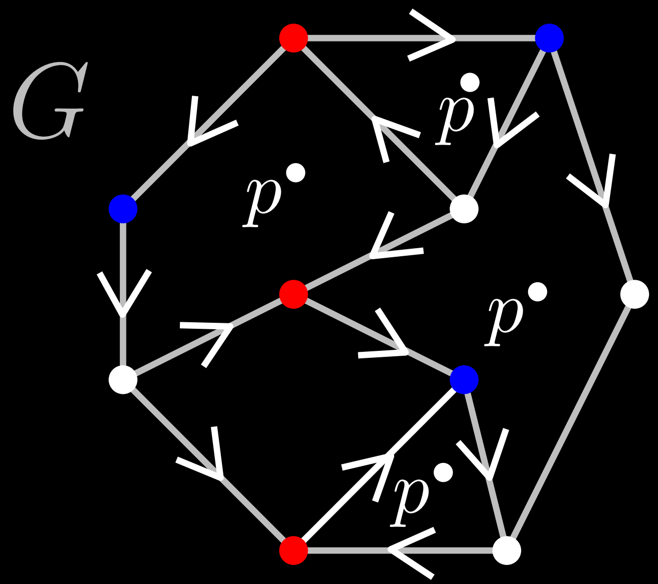

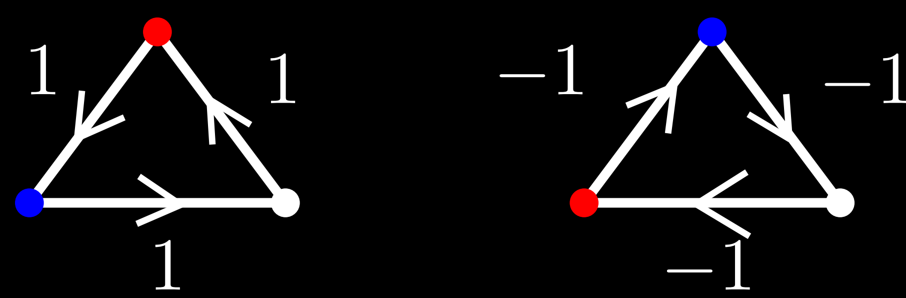

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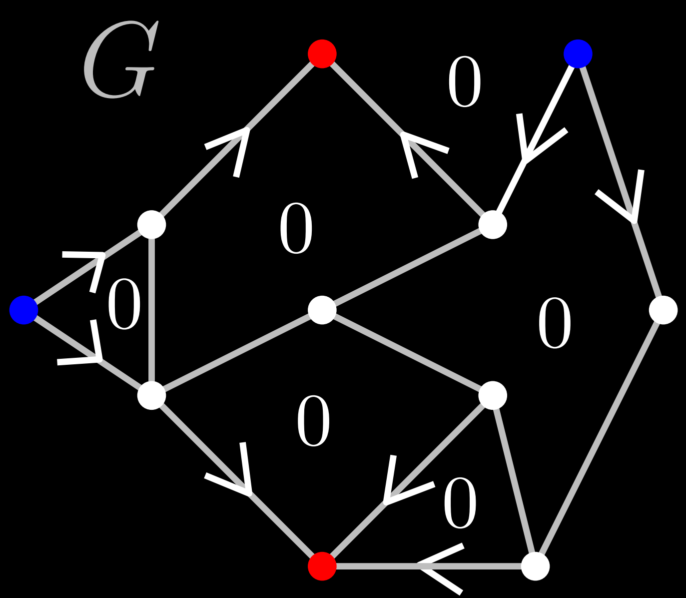

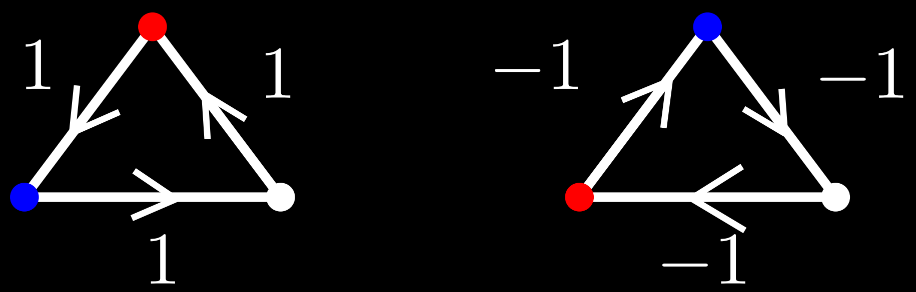

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Algorithm: Check the necessary winding number condition.
Hence, after some preprocessing we assume that
(i) winding number condition holds;
(ii)V_i's form independent sets;
(iii) G is connected; and
(iv) faces are "simple".
Solve a perfect matching problem. $\square$

C-planarity of Emedded Cyclic c-Graphs

Simple faces:
(i) Labels change sign at most four times in the facial walk; and
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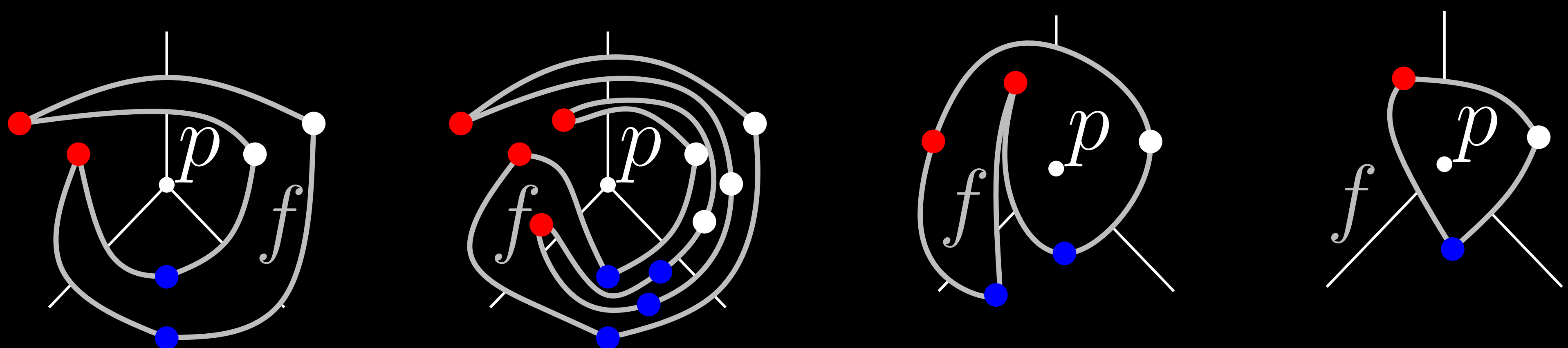

+1,+1,+1,-1,-1,-1

+1,+1,+1,-1,-1,-1,
+1,+1,+1,-1,-1,-1

+1,-1,+1,+1,+1

+1,+1,+1

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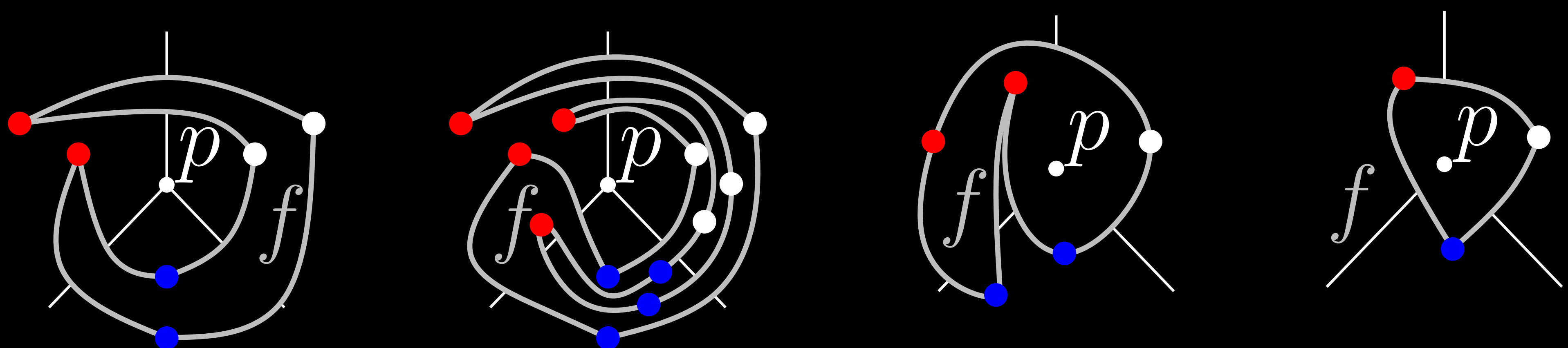

+1,+1,+1,-1,-1,-1

+1,+1,+1,-1,-1,-1,
+1,+1,+1,-1,-1,-1

+1,-1,+1,+1,+1

+1,+1,+1

We assume that $|wn(f)| > 0$ for exactly one inner face of G.

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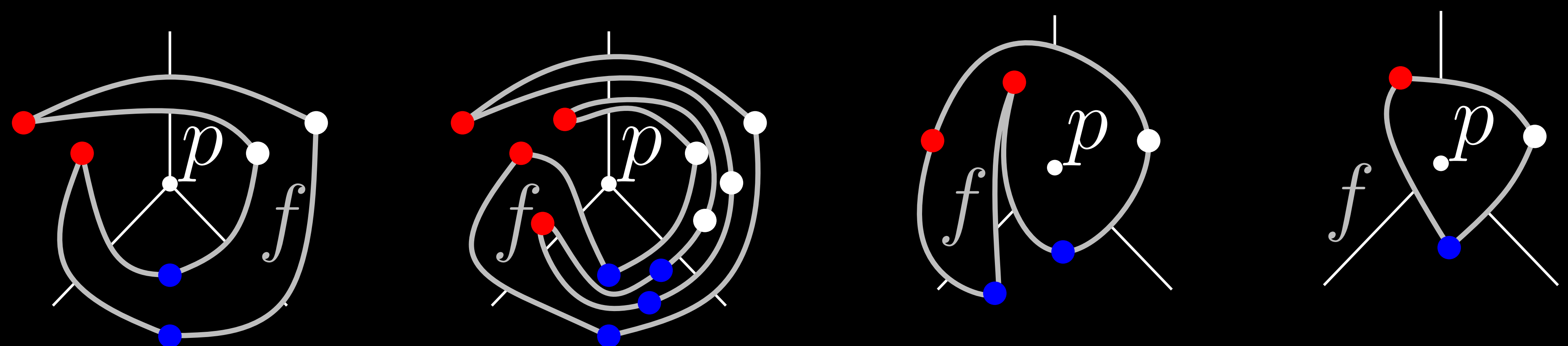

+1,+1,+1,-1,-1,-1

+1,+1,+1,-1,-1,-1, +1,+1,+1,-1,-1,-1

+1,-1,+1,+1,+1

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Otherwise, the problem was solved by Angelini et al. (2013).

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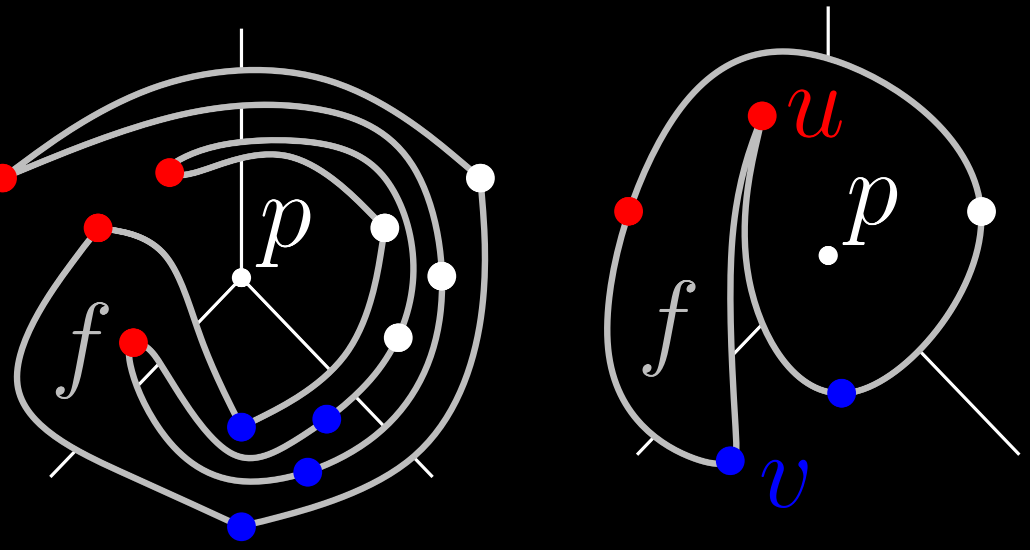

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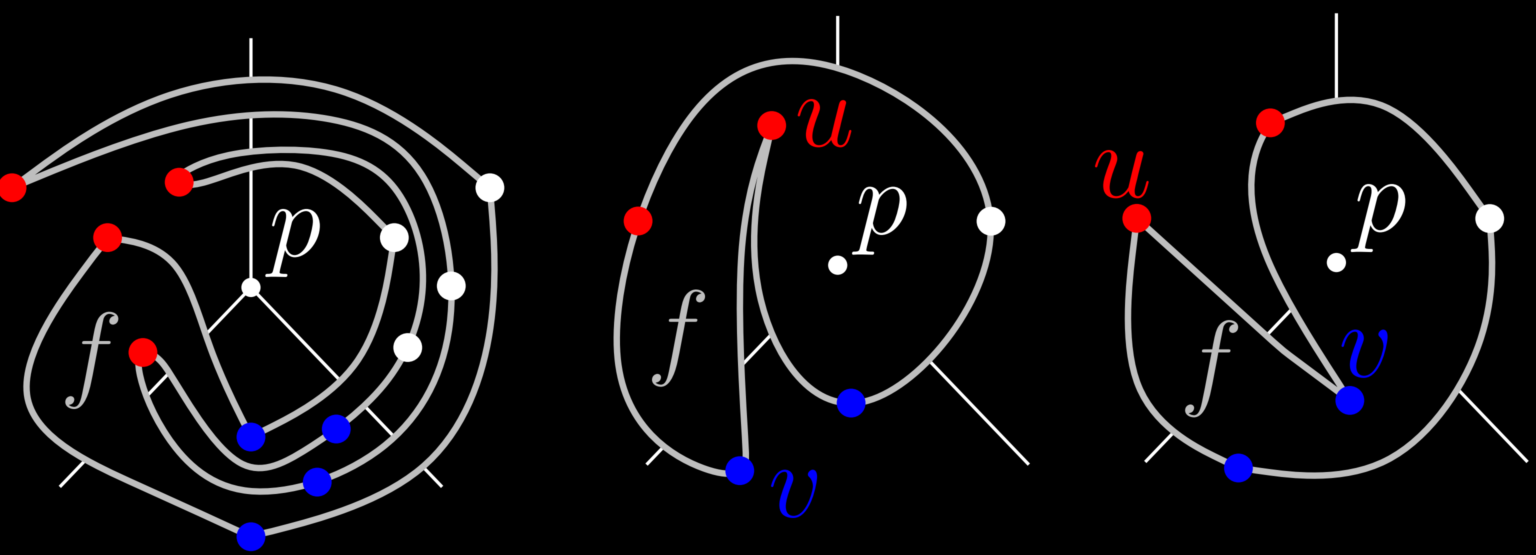

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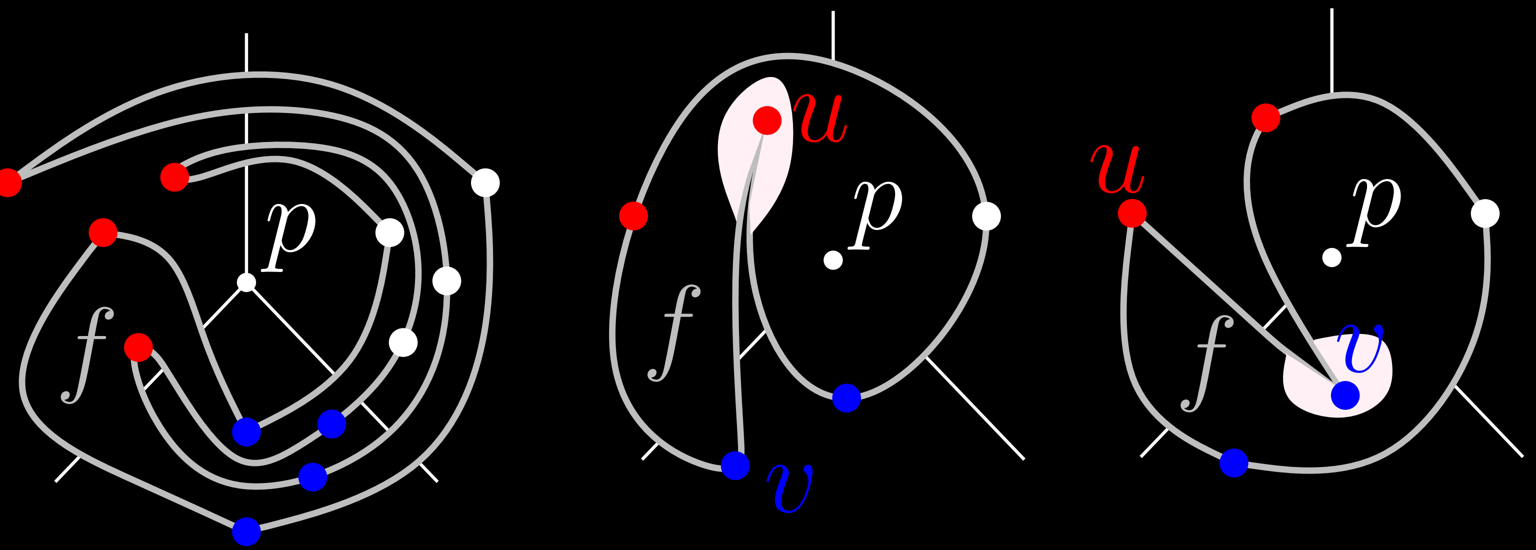

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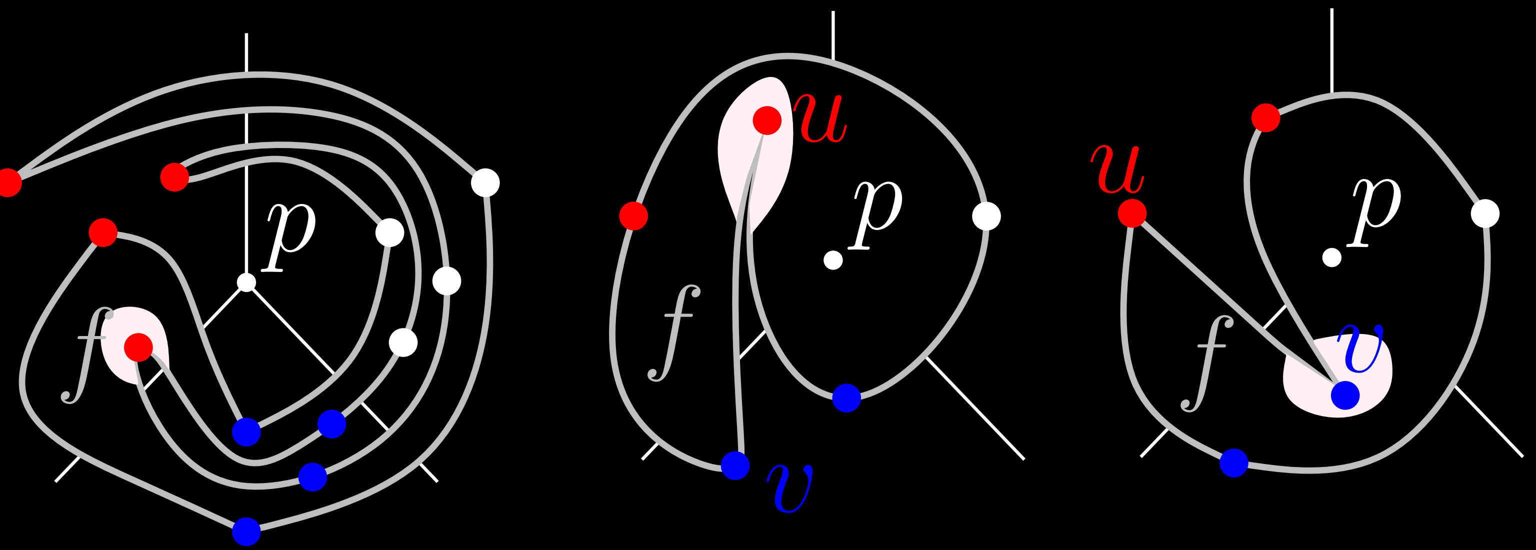

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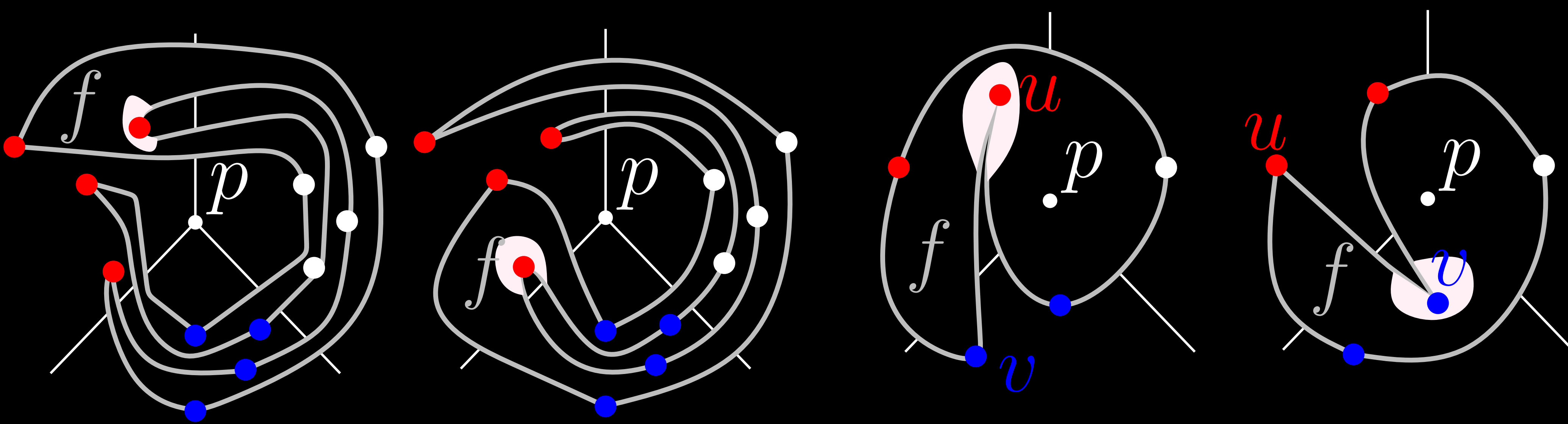

+1,+1,+1,-1,-1,-1,
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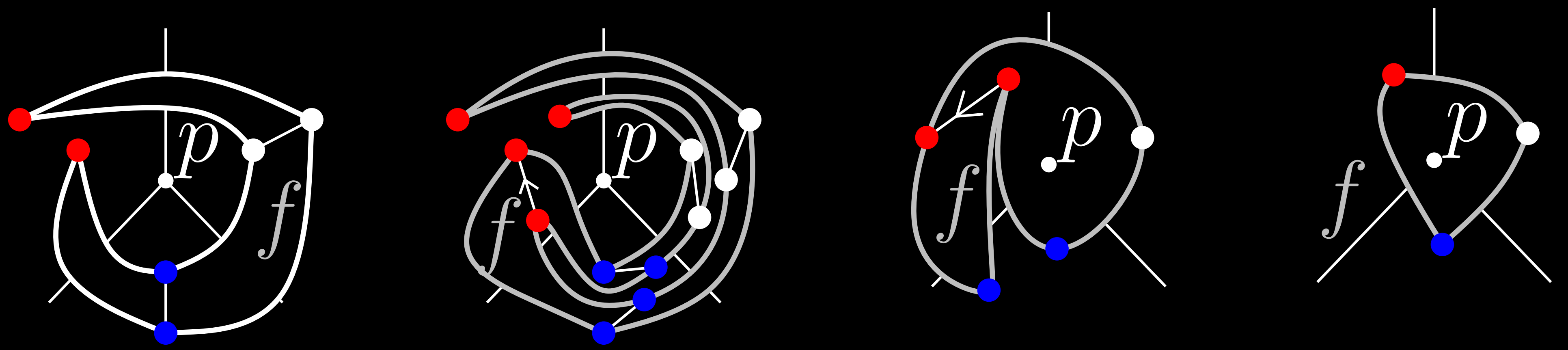

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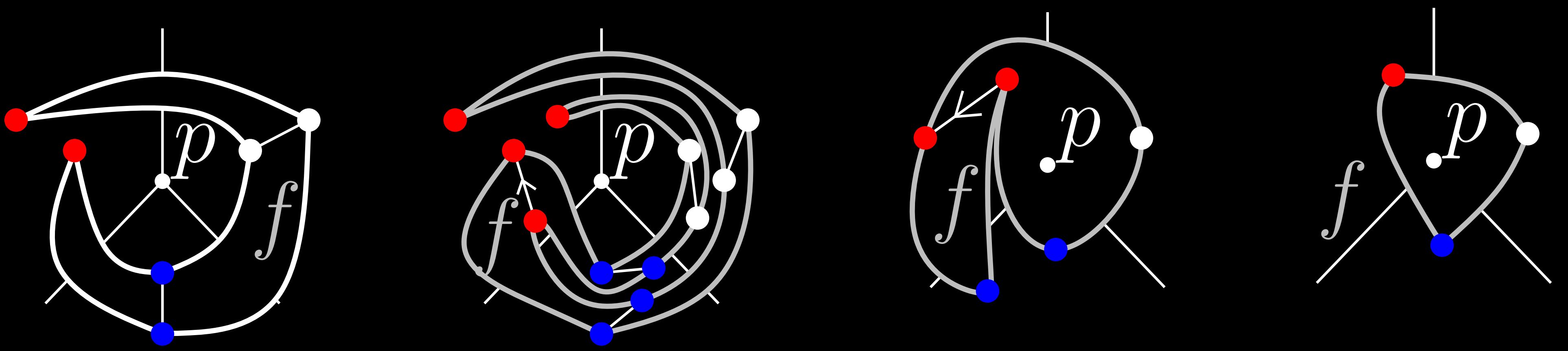

The crux is to prove that we do not obtain a directed cycle.

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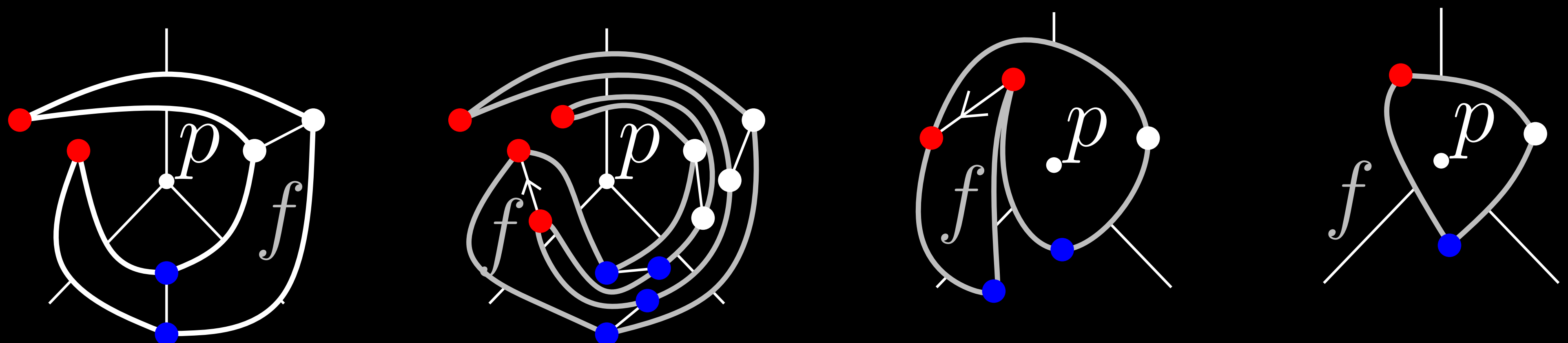

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Main observation: An area-minimal directed cycle $\overrightarrow{C}$ contains a cycle C' such that $|wn(C')| = 1$ in the cl. of its interior.

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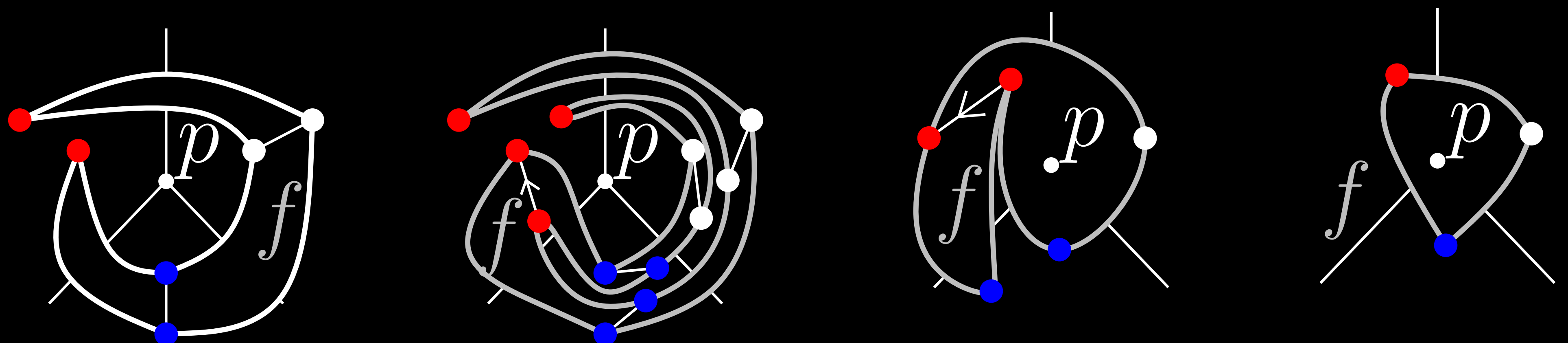

The crux is to prove that we do not obtain a directed cycle.

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$$\mathbf{0} = \mathbf{wn}(\overrightarrow{C}) = \sum_{\mathbf{f} \subset \mathbf{int}(\overrightarrow{C})} \mathbf{wn}(\mathbf{f}) = \mathbf{wn}(\mathbf{C}') + \mathbf{0} \neq \mathbf{0}$$

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Assuming that we can decide if a map is approximable in a polynomial time. Can we get FPT of c-planarity w.r.t. to the number of faces of the input graph?

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That would imply FTP w.r.t. the number of faces of both the input graph and the target graph.