Non-aligned drawings of planar graphs

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Graph Drawing
September 19, 2016

Non-aligned drawings

Drawing graphs with placement of vertices allowing fast operations

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A non-aligned drawing of a graph with n vertices is:

- on an $f(n) \times g(n)$ grid, for some functions f and g
- vertices at the intersection of the grid
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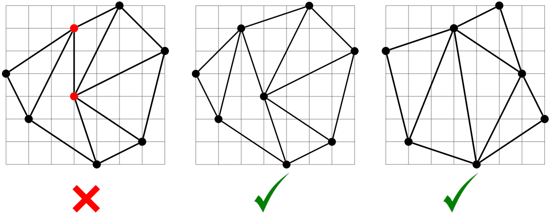

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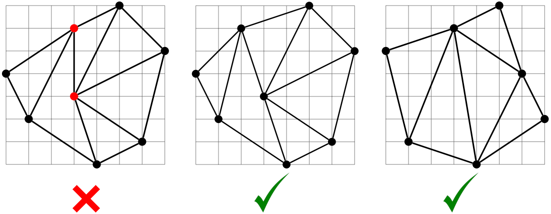

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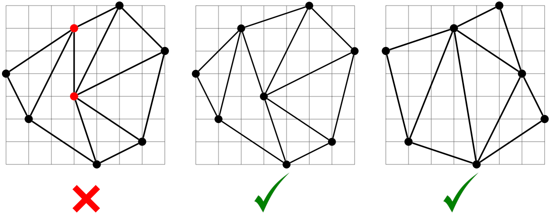

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Edges: "straight-line" or "bend" (on the grid points)

Our results

Every planar graph with n vertices has a:

- non-aligned drawing in a $n \times n$-grid with $\leq \frac{2n-5}{3}$ bends.
  (only 1 if the graph is 4-connected)
- non-aligned straight-line drawing in an $n \times O(n^3)$ grid
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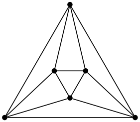

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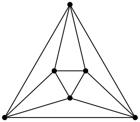

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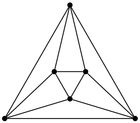

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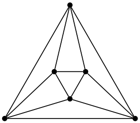

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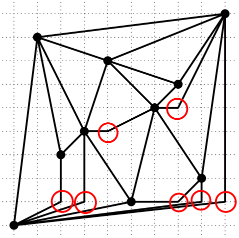

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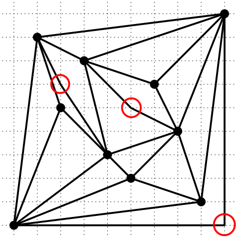

[Biedl, Pennarun '16]

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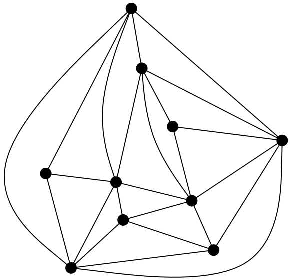
[Biedl et al. '99] If G is 4-connected, and e is an outerface edge, then $G - e$ has a non-aligned drawing on an $n \times n$ grid.



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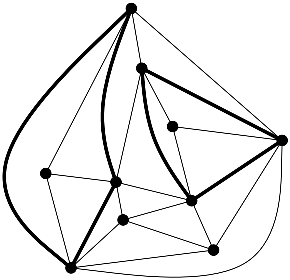

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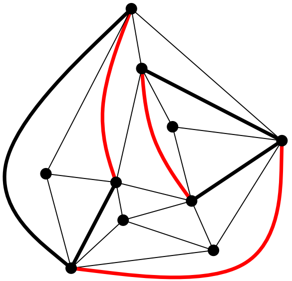

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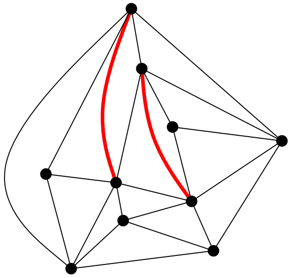

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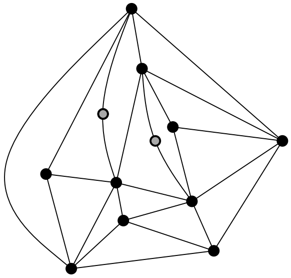

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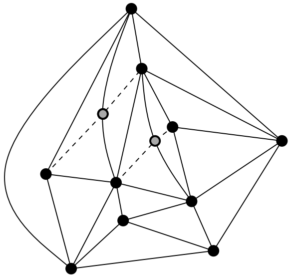

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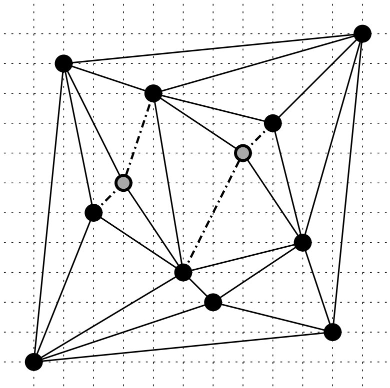

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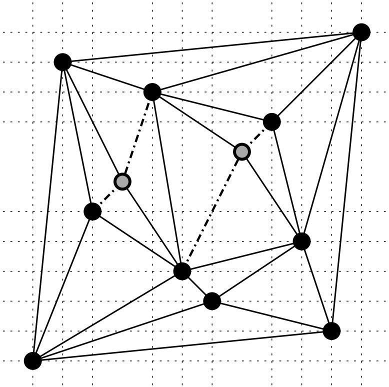
Keep relative orders, original vertices on a $n \times n$ grid and grey ones inbetween



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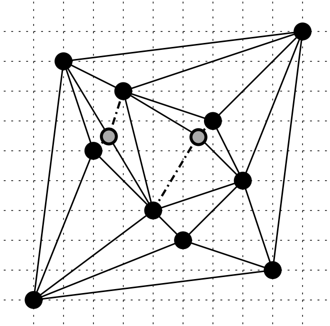

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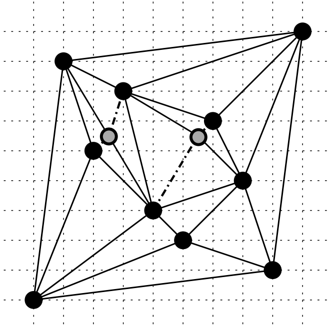

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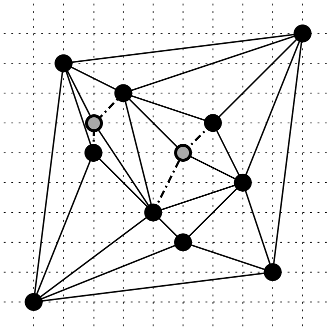

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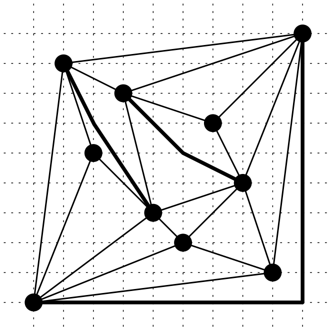

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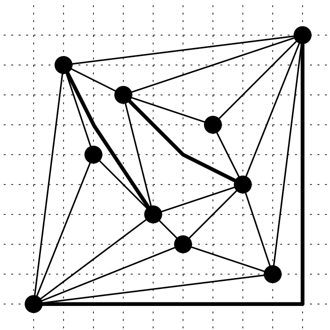

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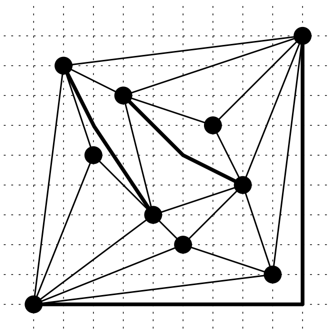

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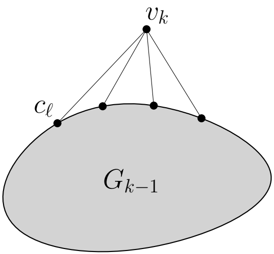
Non-aligned drawing on an $n \times O(n^3)$-grid

A canonical ordering of a maximal planar graph is a vertex order $v_1 \cdots v_n$ such that the outerface is $[v_1, v_2, v_n]$ and for any $3 \leq k \leq n$, $G_k = G[v_1 \cdots v_k]$ is 2-connected [de Fraysseix, Pach, Pollack '90].

each v_k: predecessors forming an interval on the outerface of G_{k-1}

c_ℓ: left-most predecessor



Non-aligned drawing on an $n \times O(n^3)$-grid

Topological order (based on canonical):
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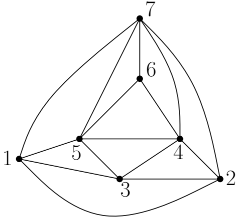

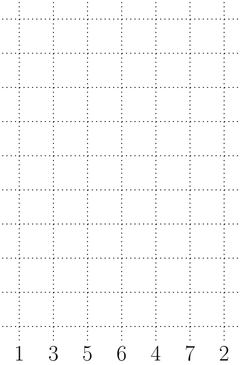

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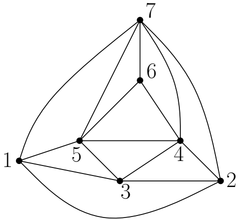

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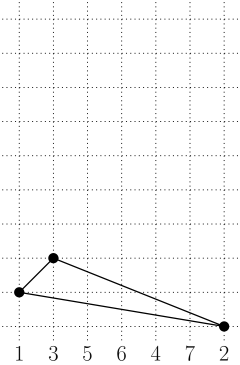

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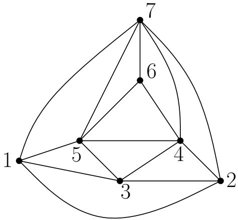

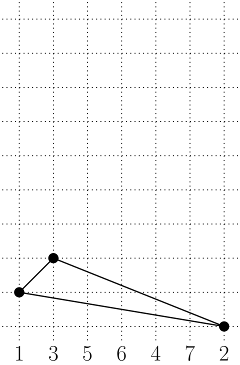

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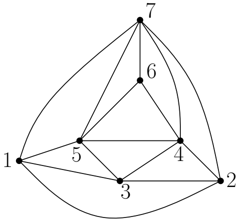

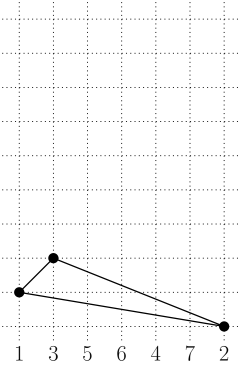

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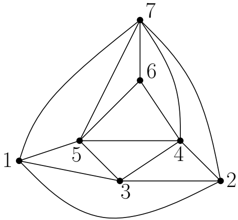

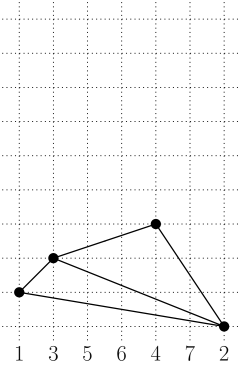

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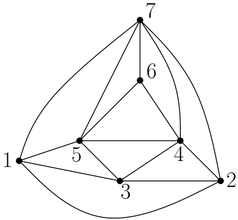

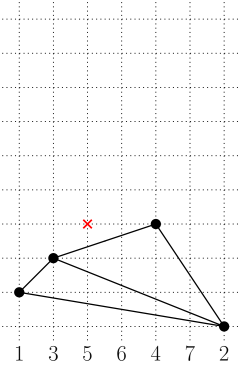

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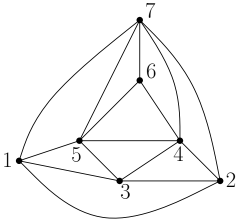

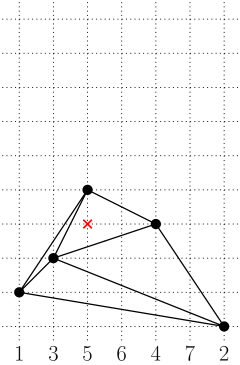

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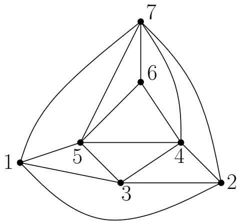

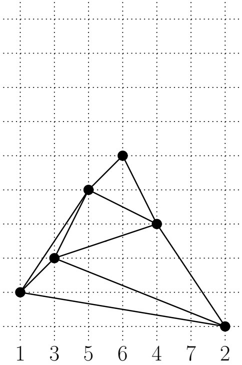

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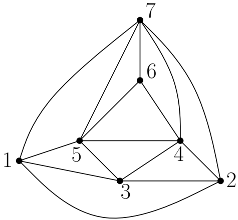
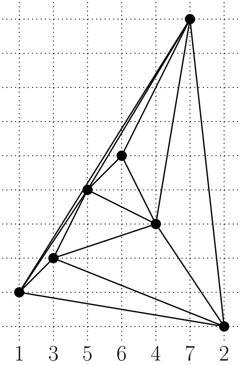
- the edge from c_ℓ has positive slope
- the row $\{y = y(v_{k+1})\}$ is empty





Non-aligned drawing on an $n \times O(n^3)$-grid

Left-steepness of a vertex: $s(v) = \left| \frac{y(v) - y(c_\ell)}{x(v) - x(c_\ell)} \right|$

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Every planar graph with n vertices has a non-aligned straight-line drawing in an $n \times \left(2 + \frac{1}{2}(n-1)(n-2)^2\right)$ grid.

And now?

Open questions:

- Find a planar graph needing more than one bend
- There is likely a better bound on the $n \times O(n^3)$ result (equation on the slopes is not tight)
- Find a planar graph needing n columns and more than $n+1$ rows

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Thank you!