Snapping Graph Drawings to the Grid Optimally

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Chair for Computer Science I:
Algorithms, Complexity and Knowledge-Based Systems
University of Würzburg

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Introduction

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  - Topological similarity (equivalence up to the collapsing of features)

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Our question:

What about topological equivalence?

Intuition

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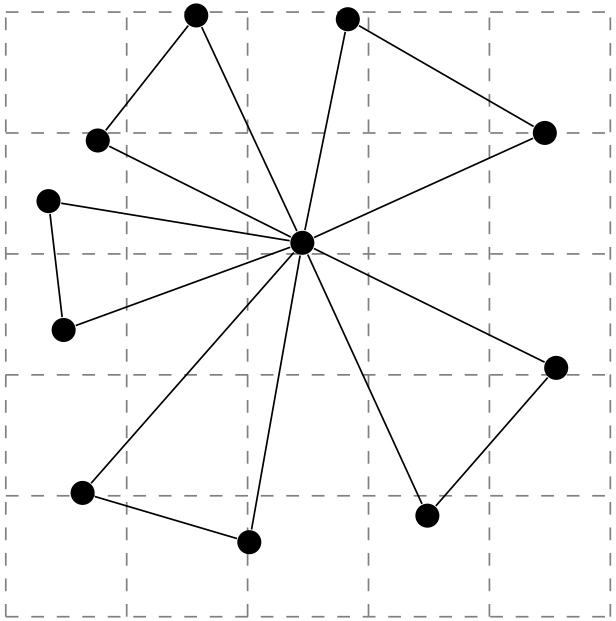

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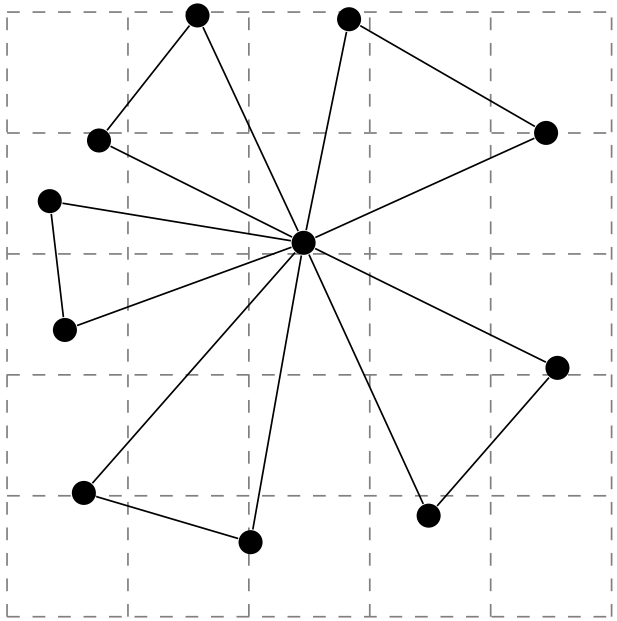

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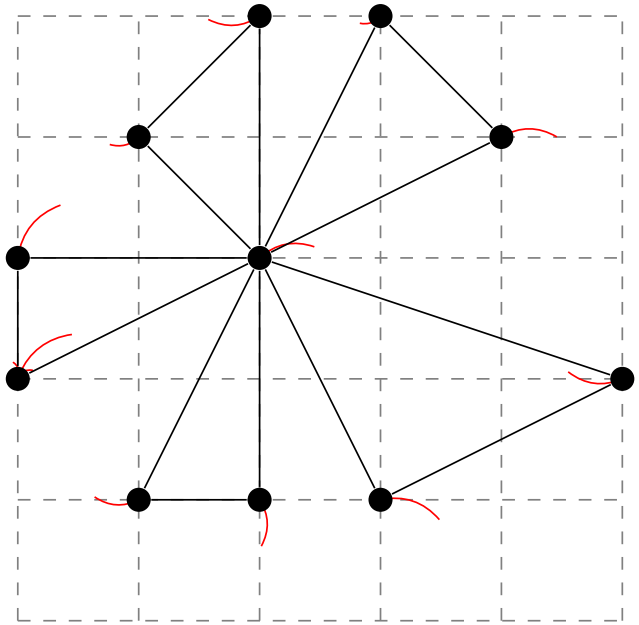

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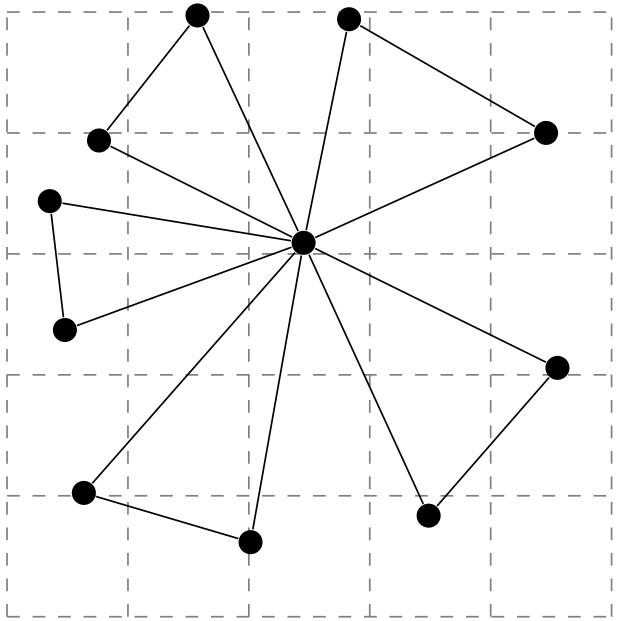

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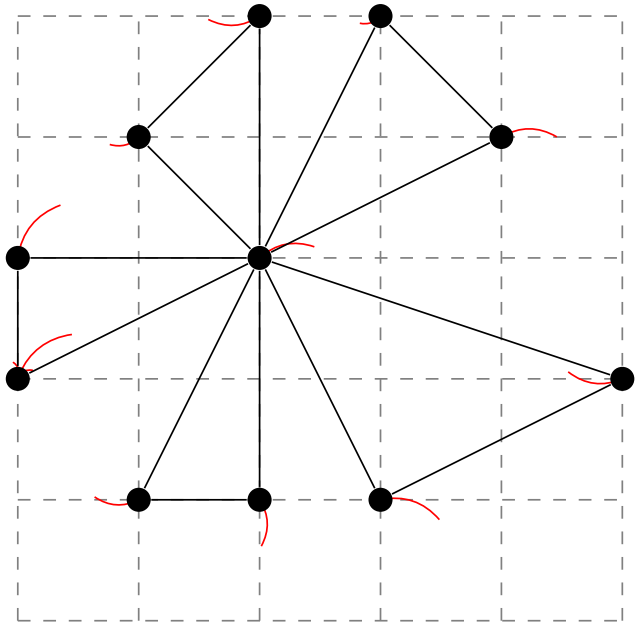

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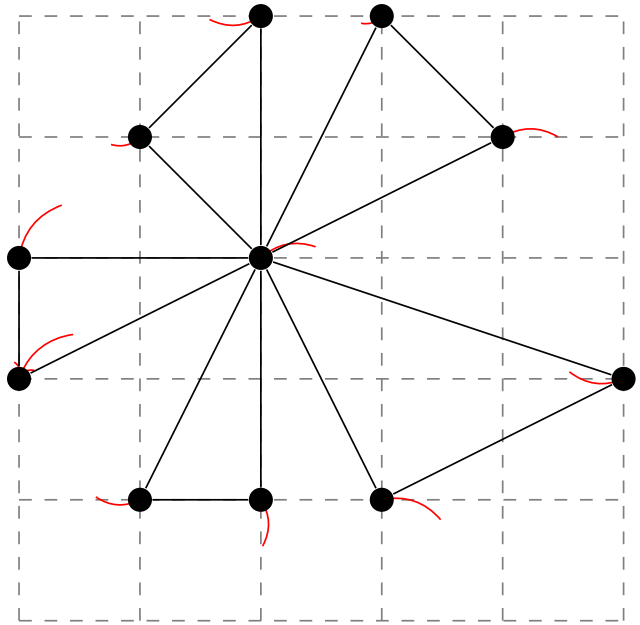

- Snap-rounding already is topologically equivalent.

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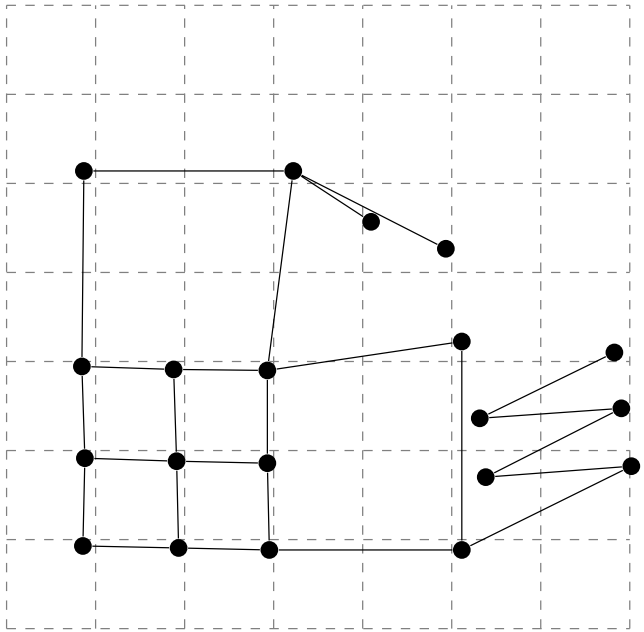

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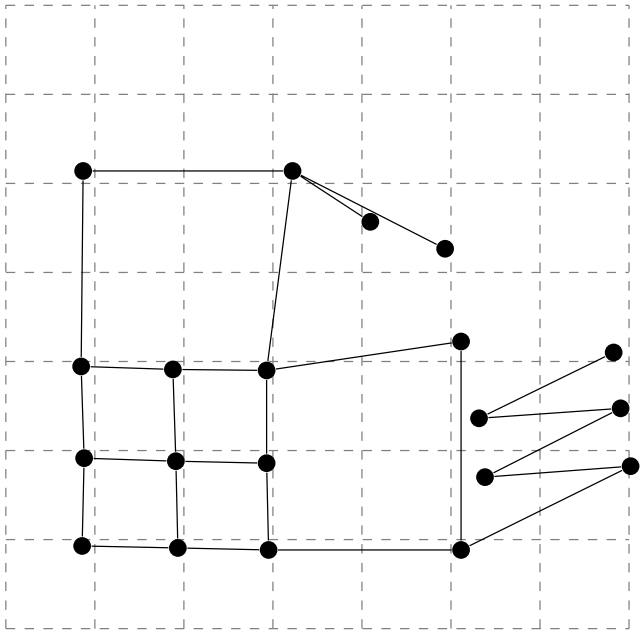

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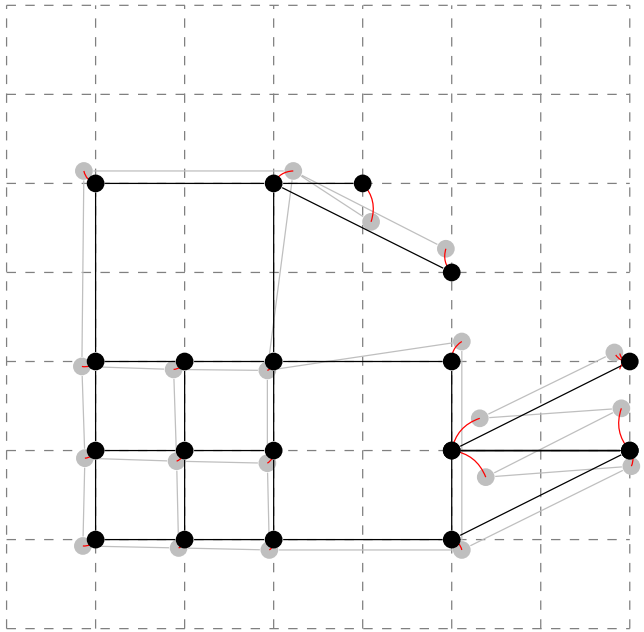

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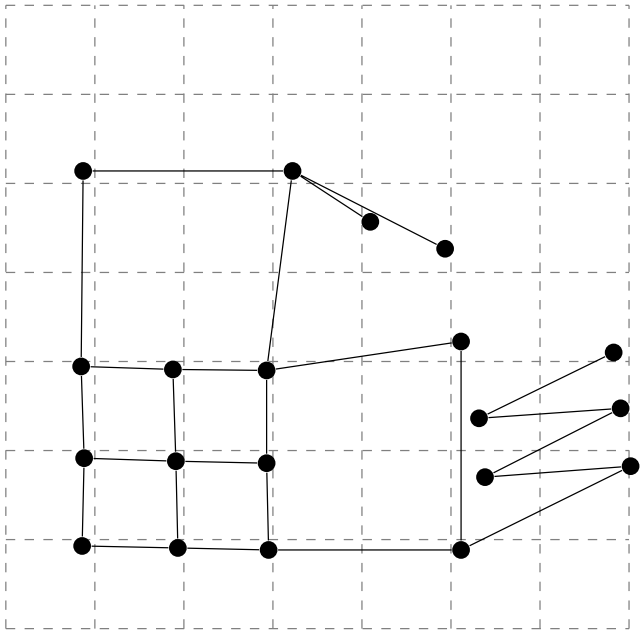

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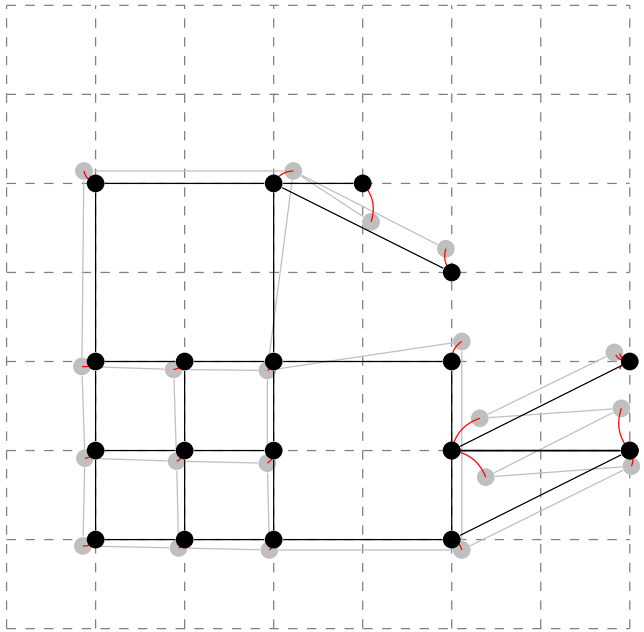

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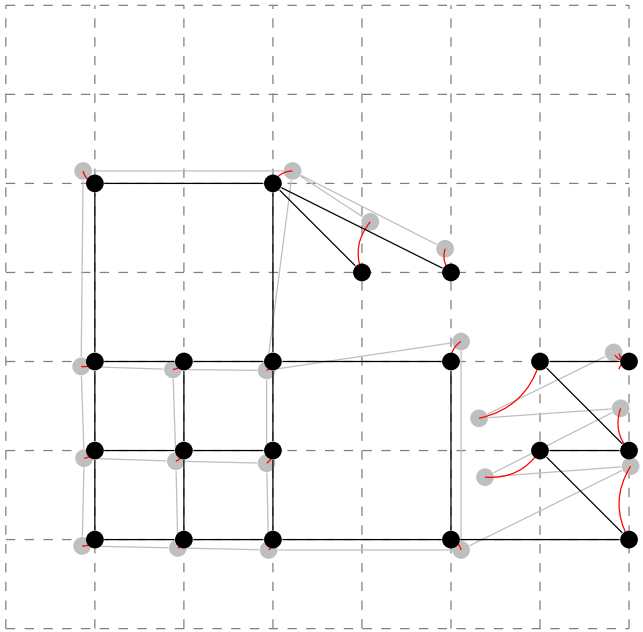

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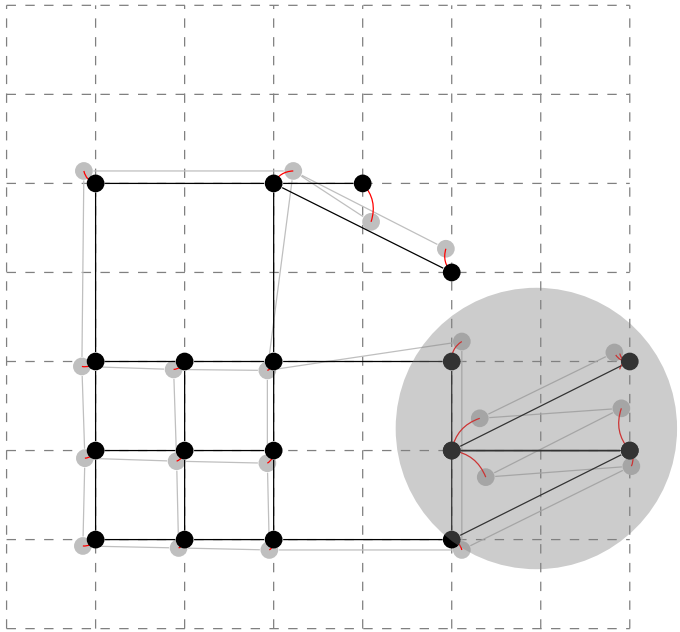

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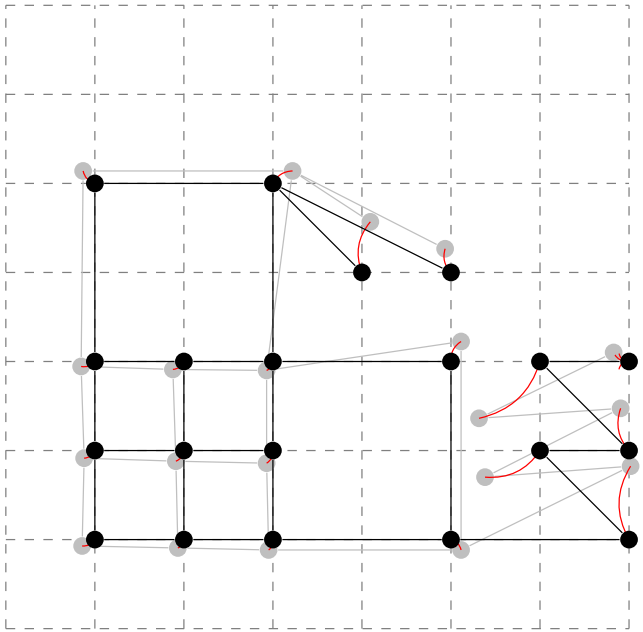

- Snap-rounding alters incidences and forces edges to collapse.

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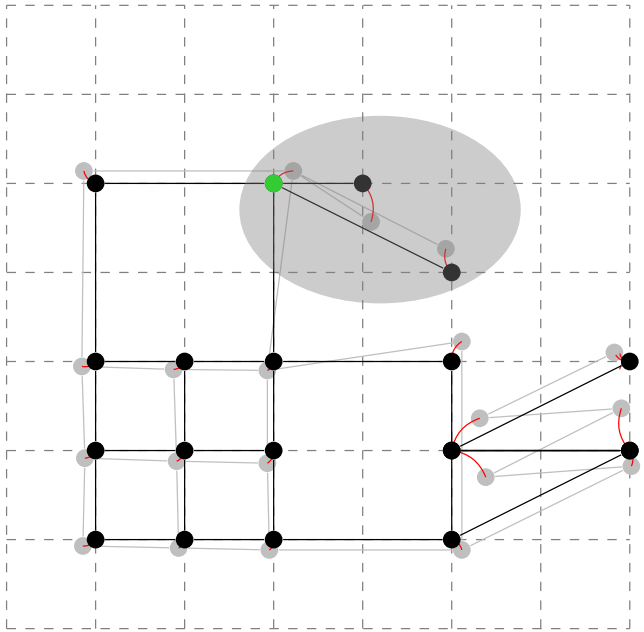

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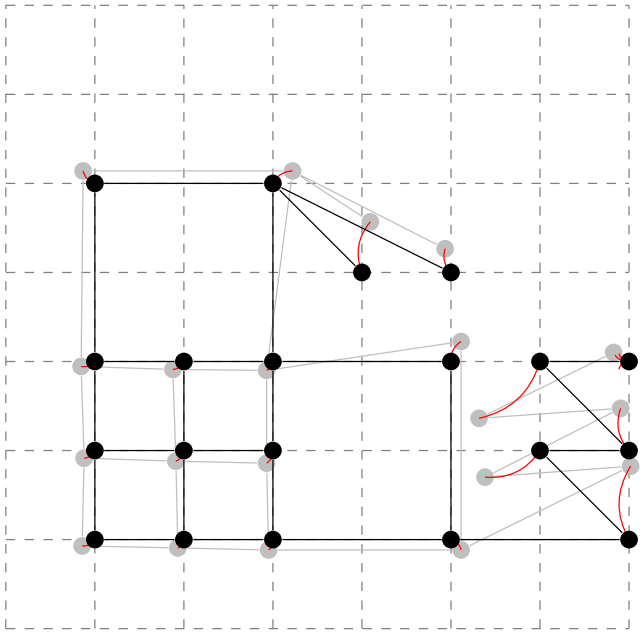

- Snap-rounding alters incidences and forces edges to collapse.
- Rounding to the nearest grid point changes the embedding of the upper-left vertex.

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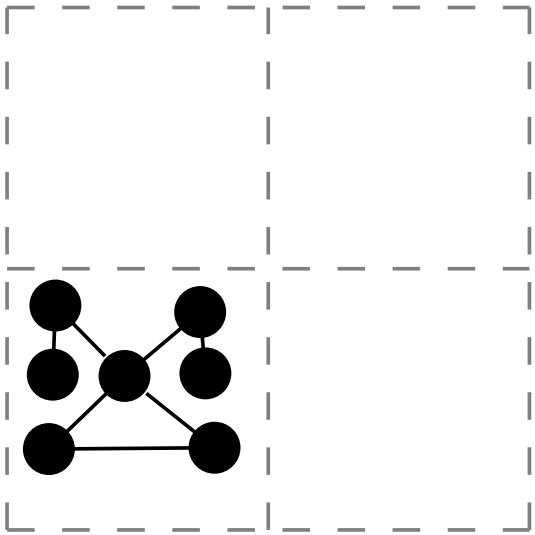

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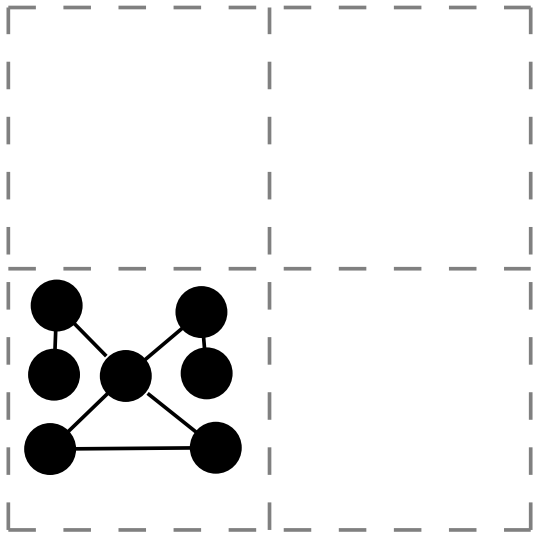

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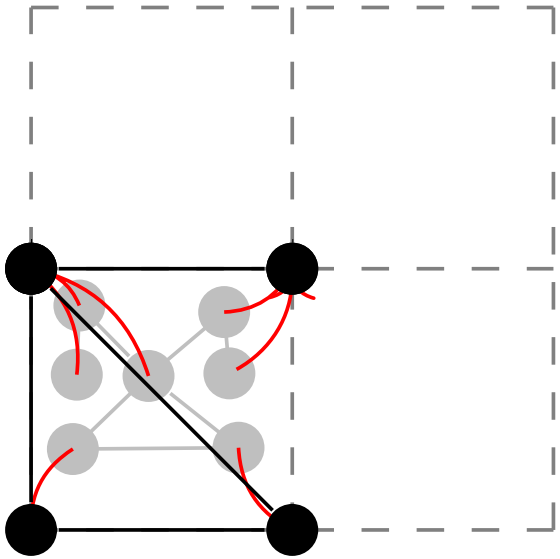

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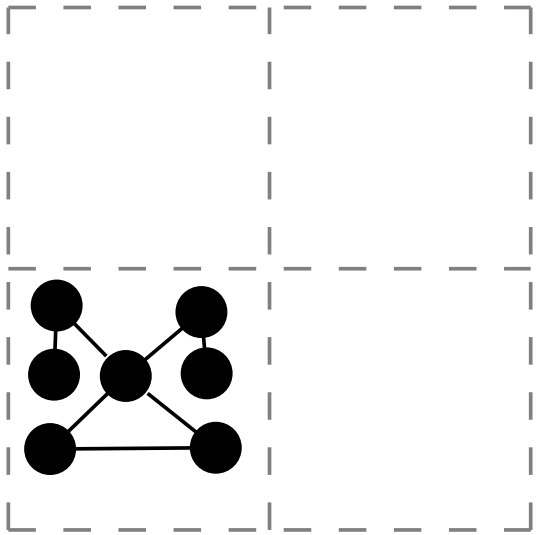

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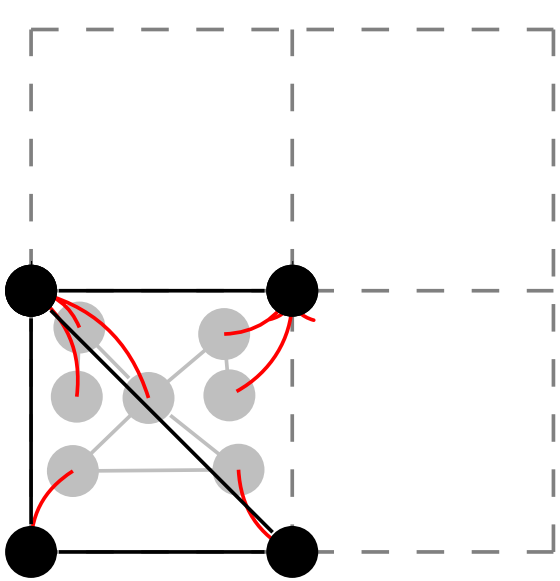

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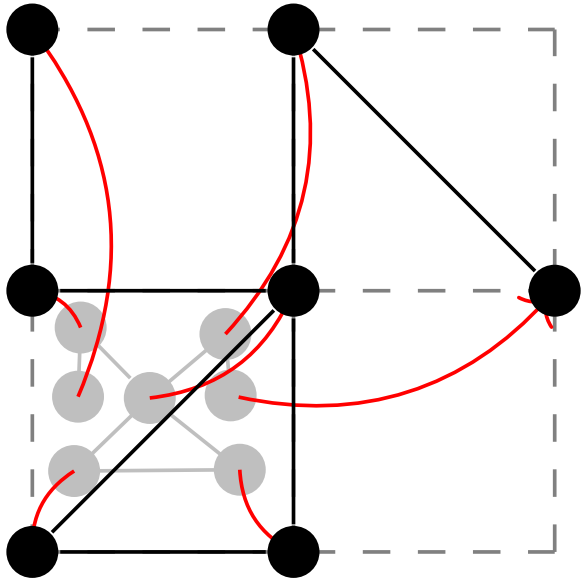

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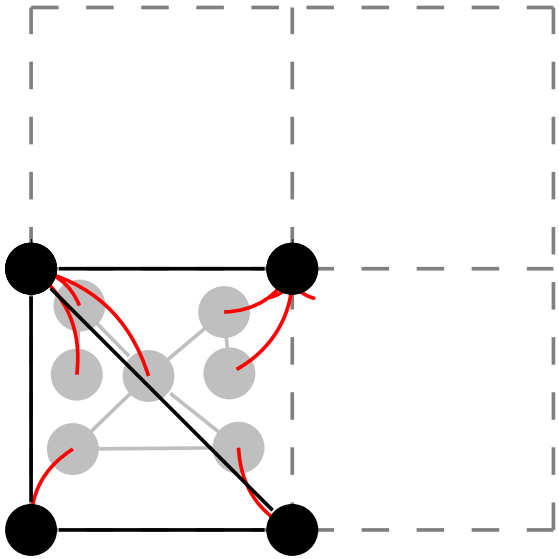

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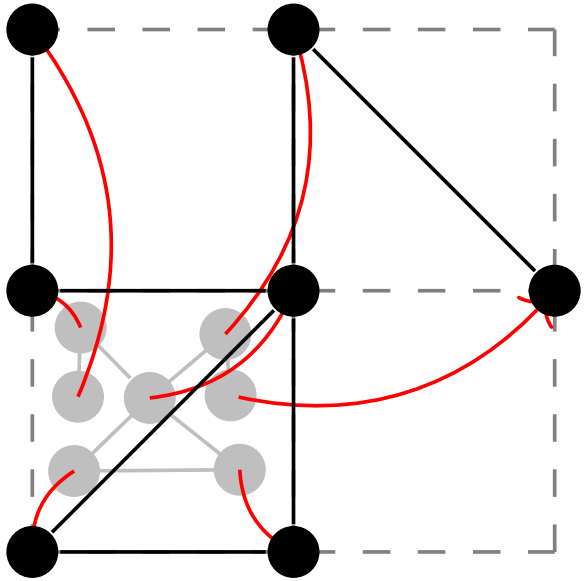

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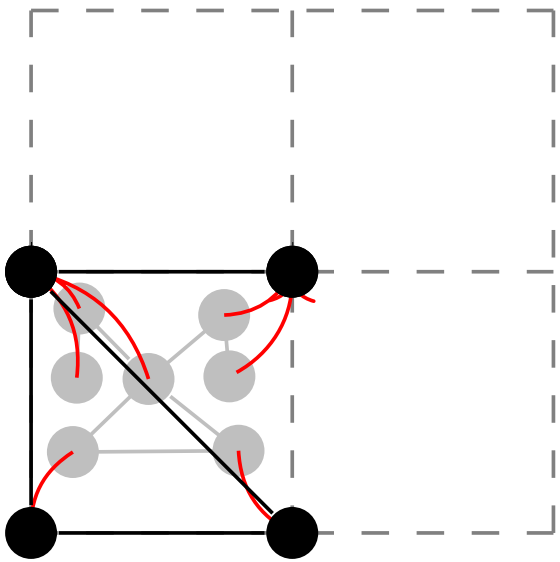

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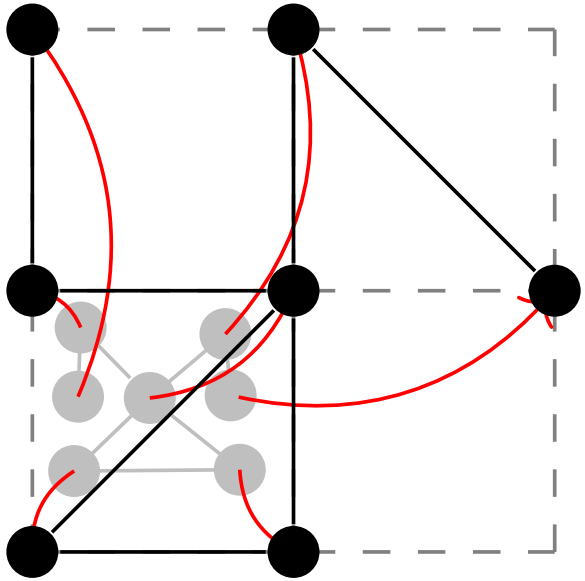

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- "Rounding" dense structures with no features collapsing is closely related to creating minimum-area drawings.

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Problem (Topologically Safe Snapping)

Graph $G = (V, E)$ with given embedding, bounding box $B = [0, X_{\max}] \times [0, Y_{\max}]$.

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- Movement is measured in Manhattan-distance.

Our Contribution

- $\mathcal{NP}$-hardness proof for Topologically Safe Snapping.

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- Experimental Evaluation

The $\mathcal{NP}$-hardness proof

NP-Hardness

- We reduce from Planar Monotone 3SAT.

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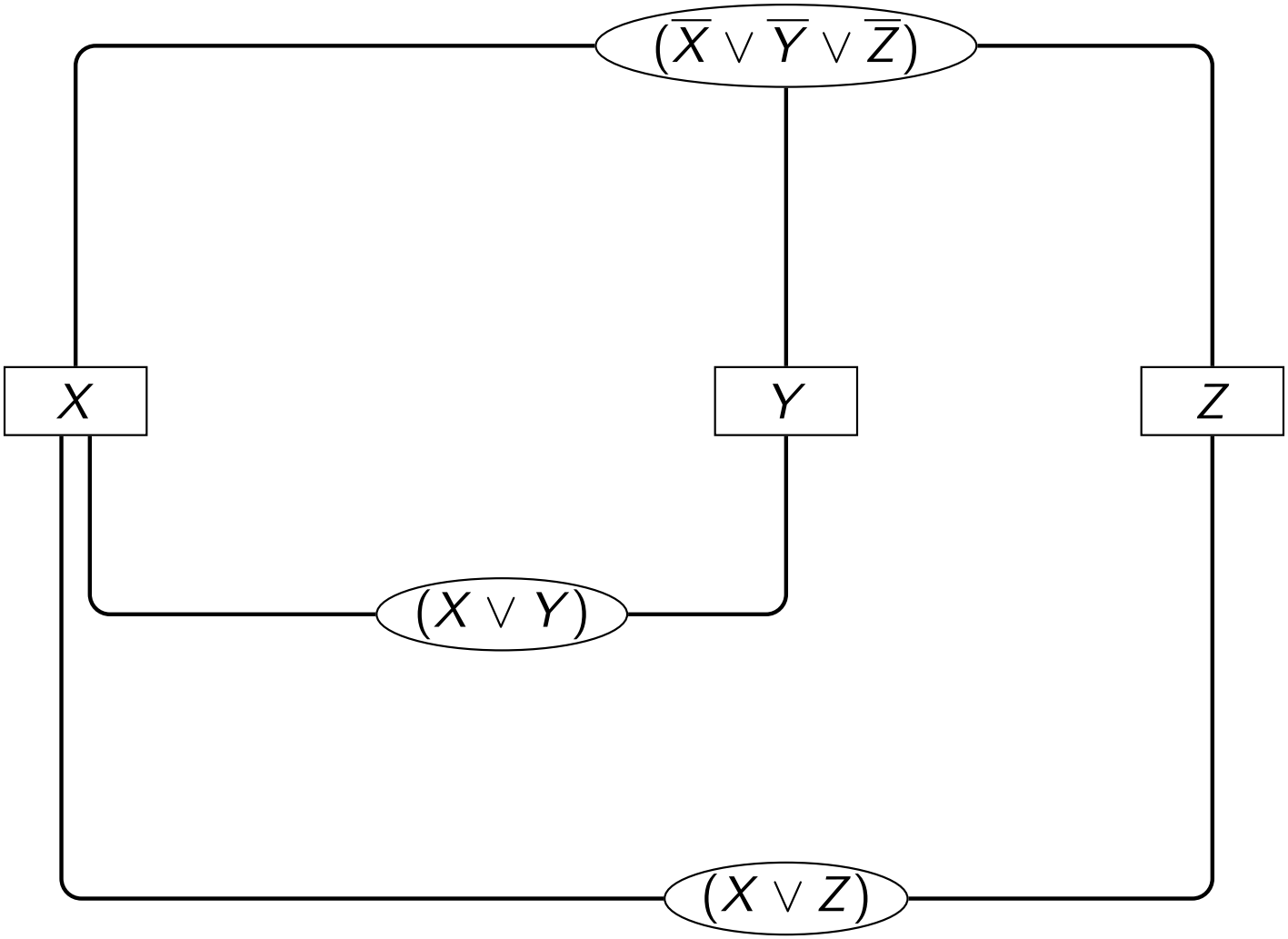
- For reduction, consider a decision variant:

Problem (Cost-bound Topologically Safe Snapping)

Graph $G = (V, E)$ with given embedding,
bounding box $B = [0, X_{\max}] \times [0, Y_{\max}]$, cost-bound $c_{\min} \in \mathbb{R}^+$.

Can G be rounded to integer coordinates within B, preserving the given embedding with total movement of $c_{\min}$?

PM3SAT-formula



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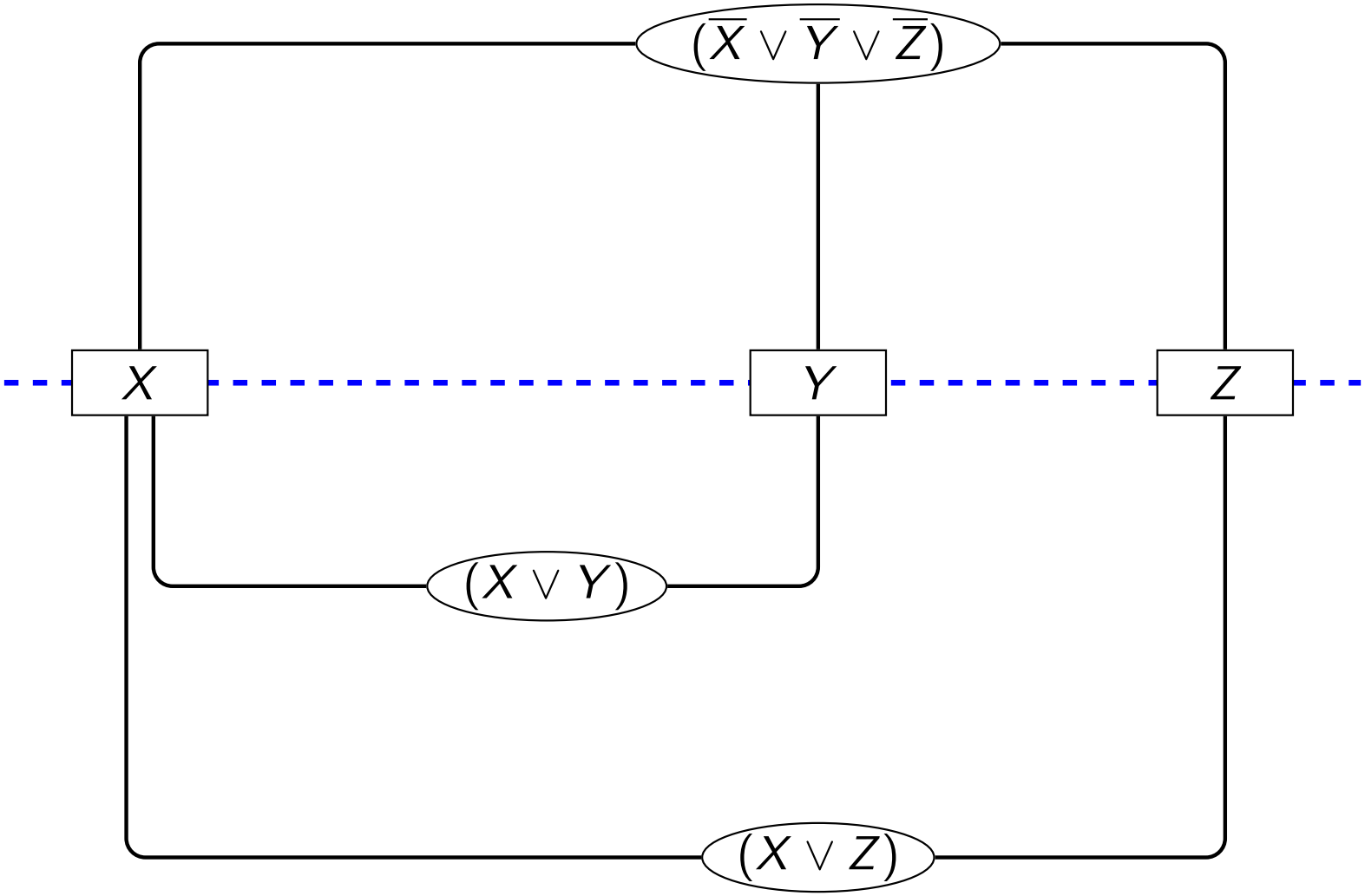

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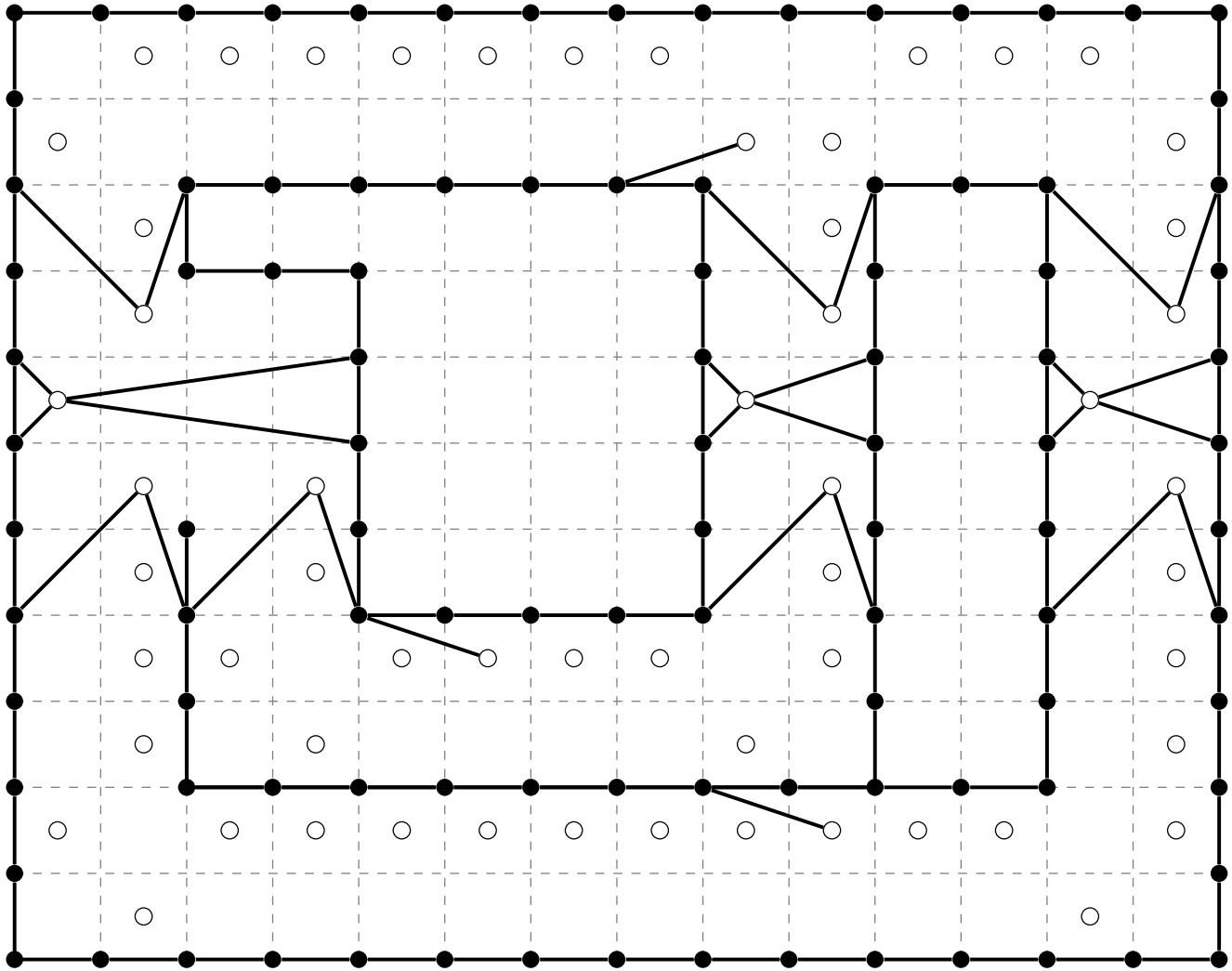

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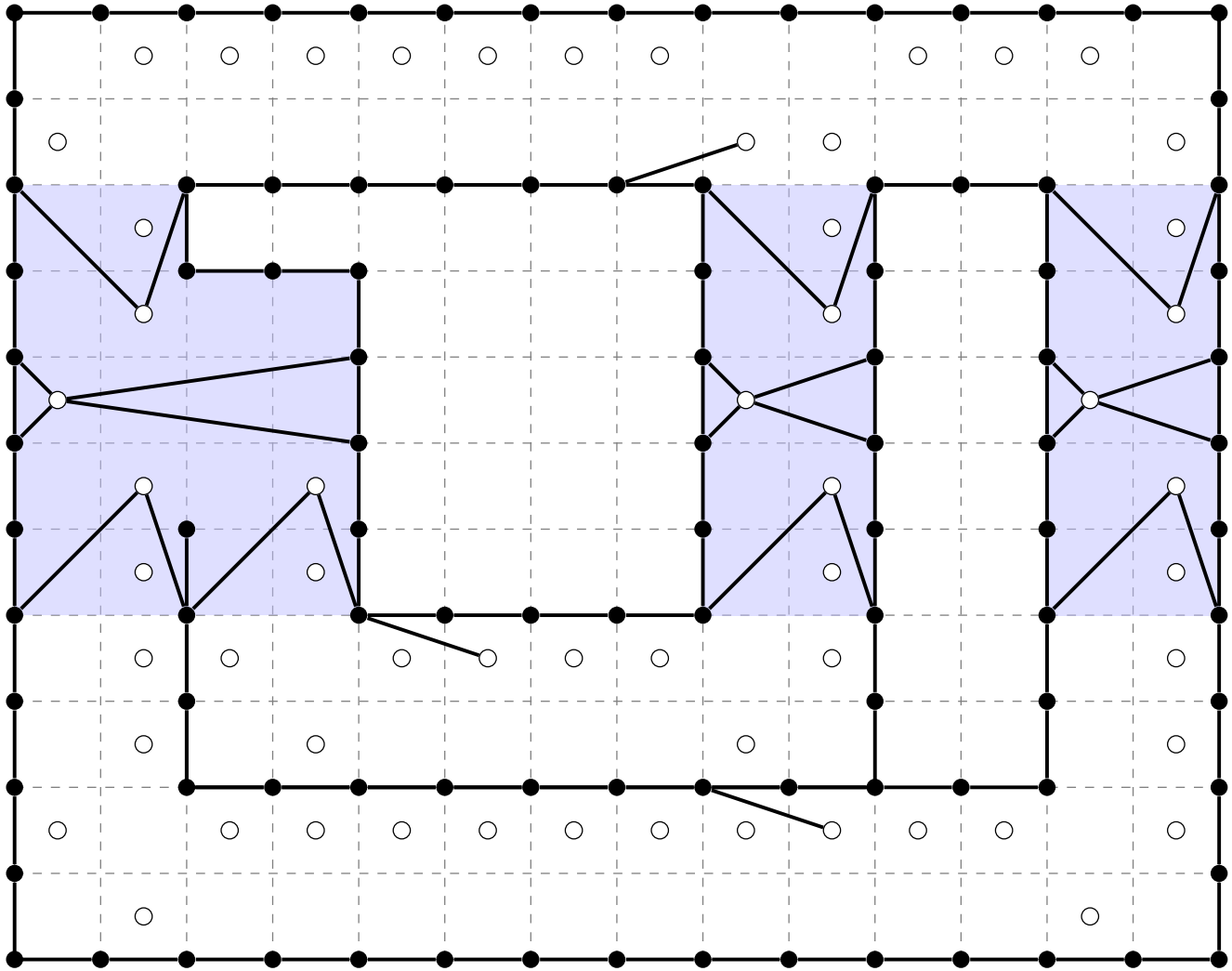

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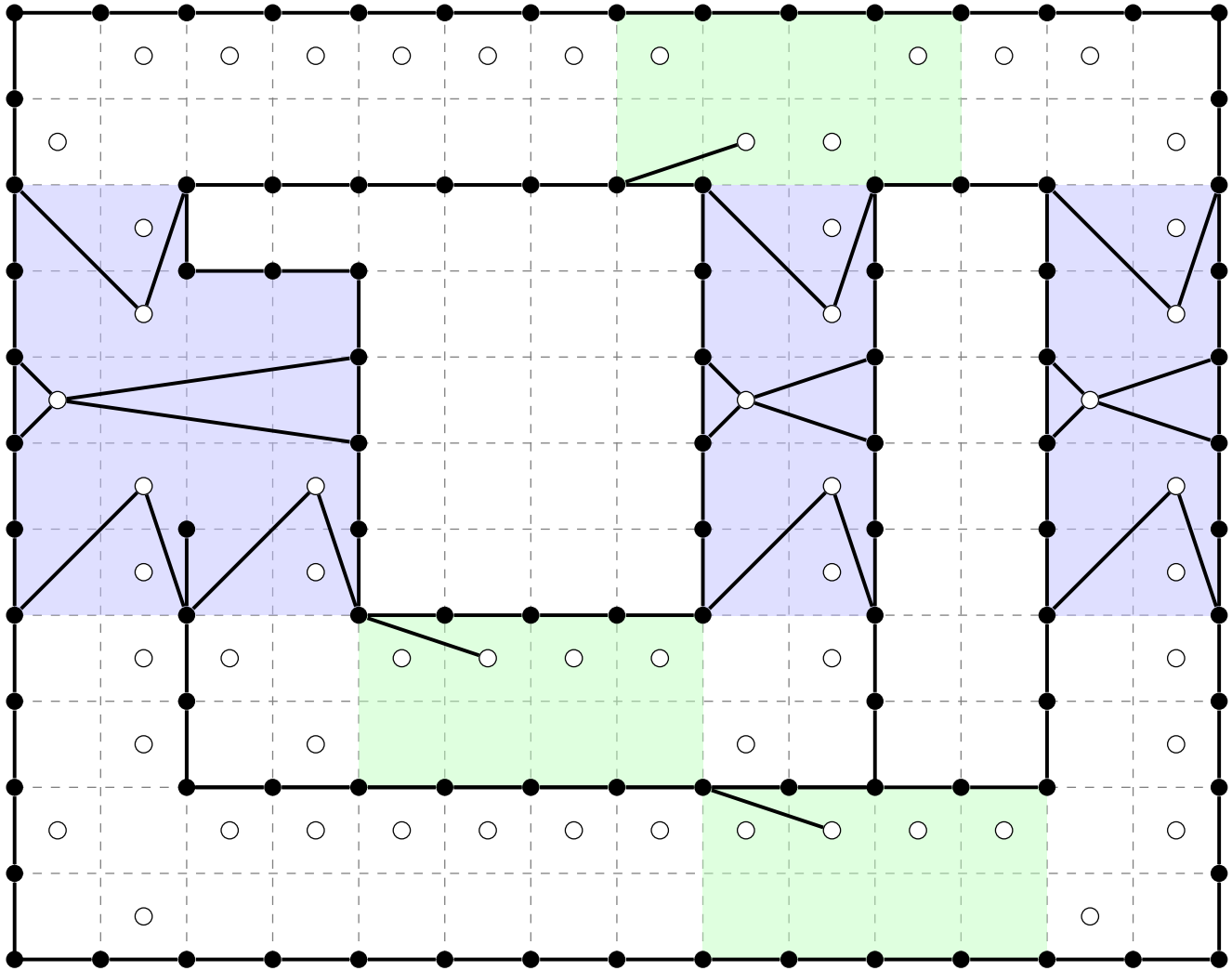

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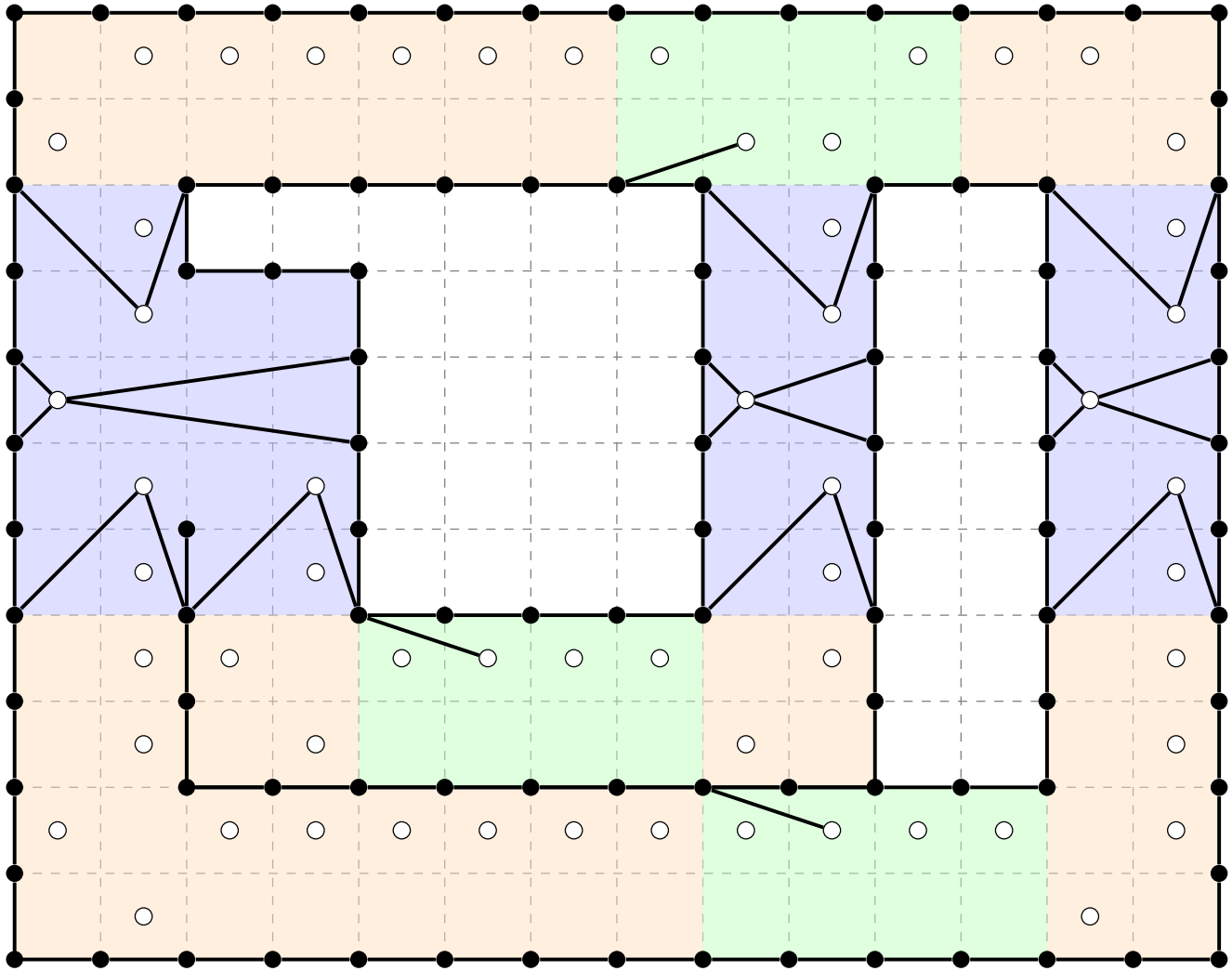

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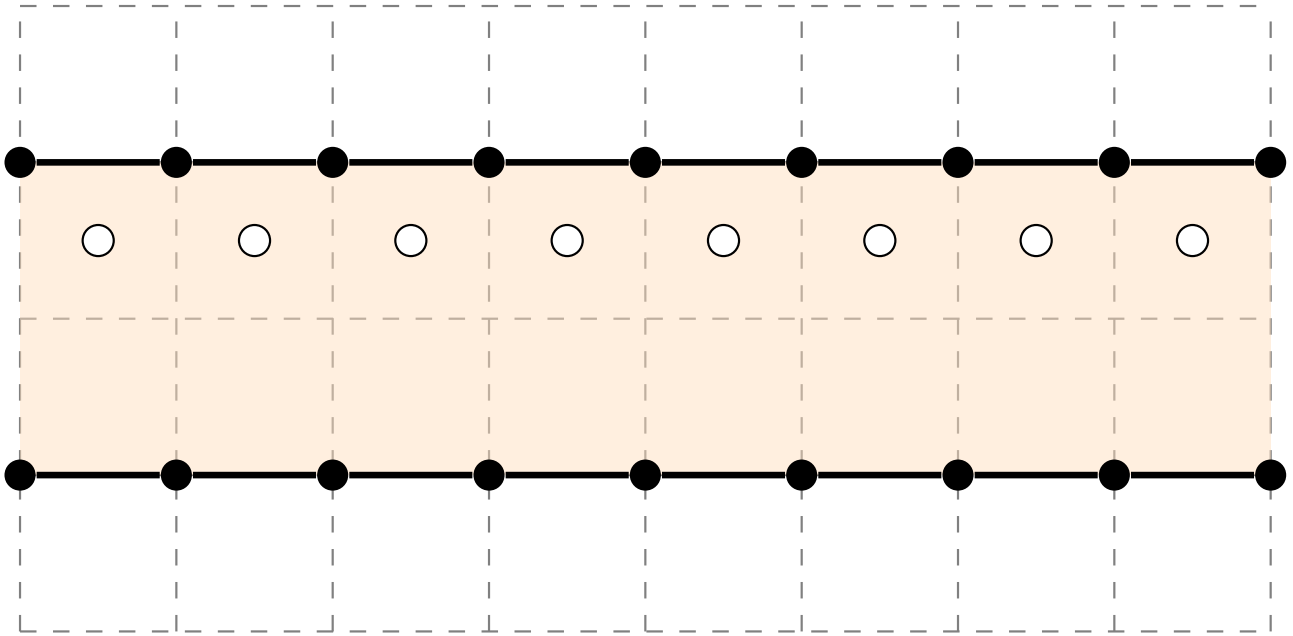

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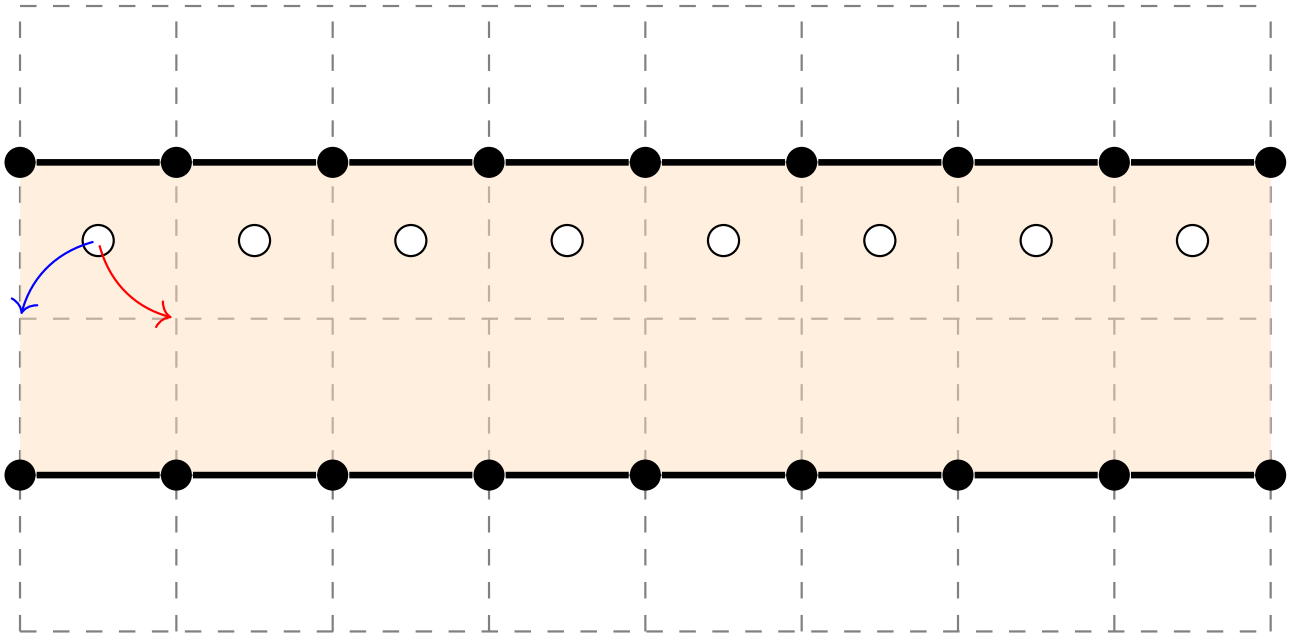

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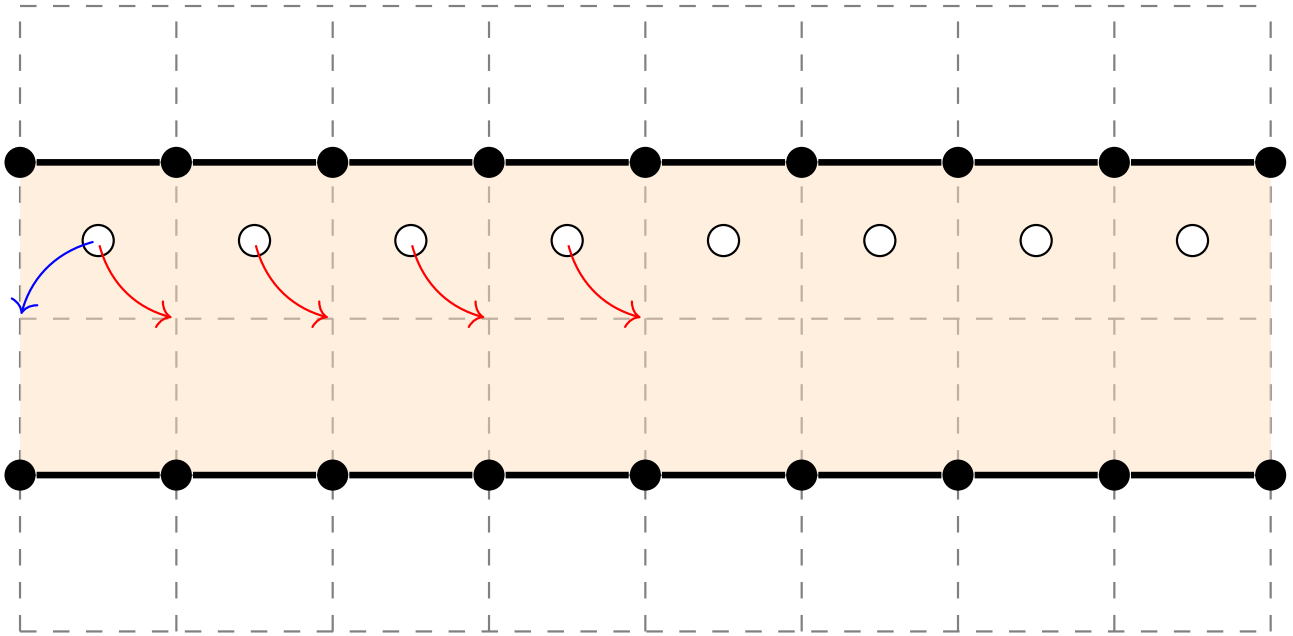

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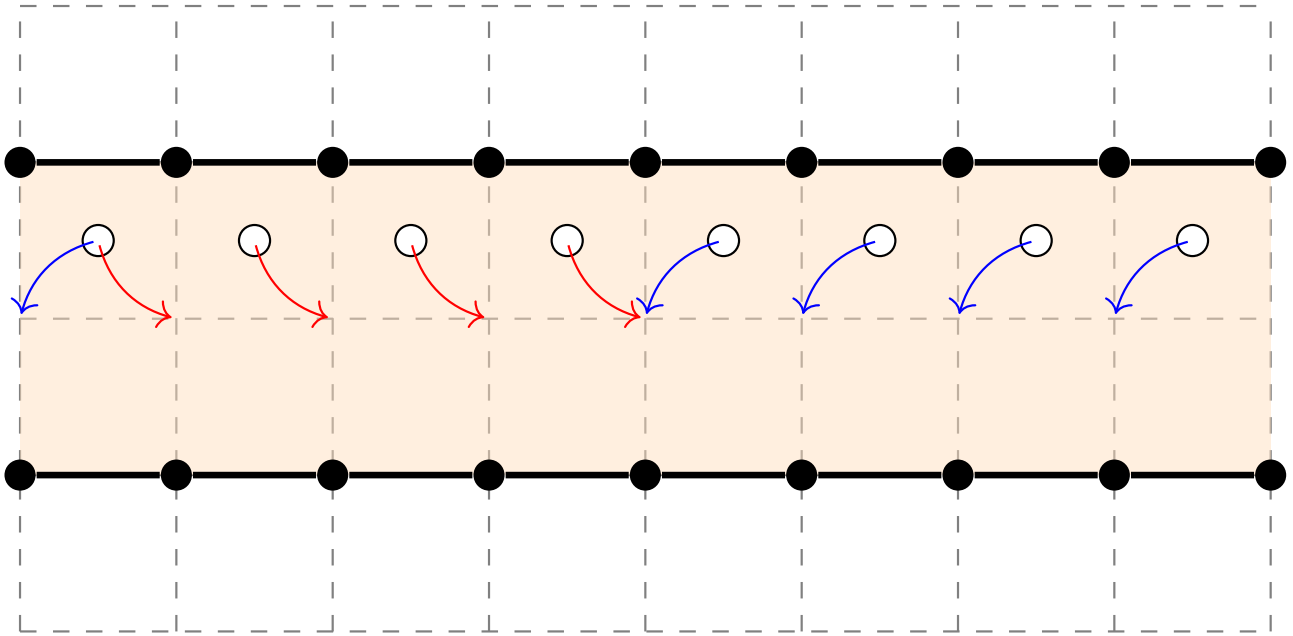

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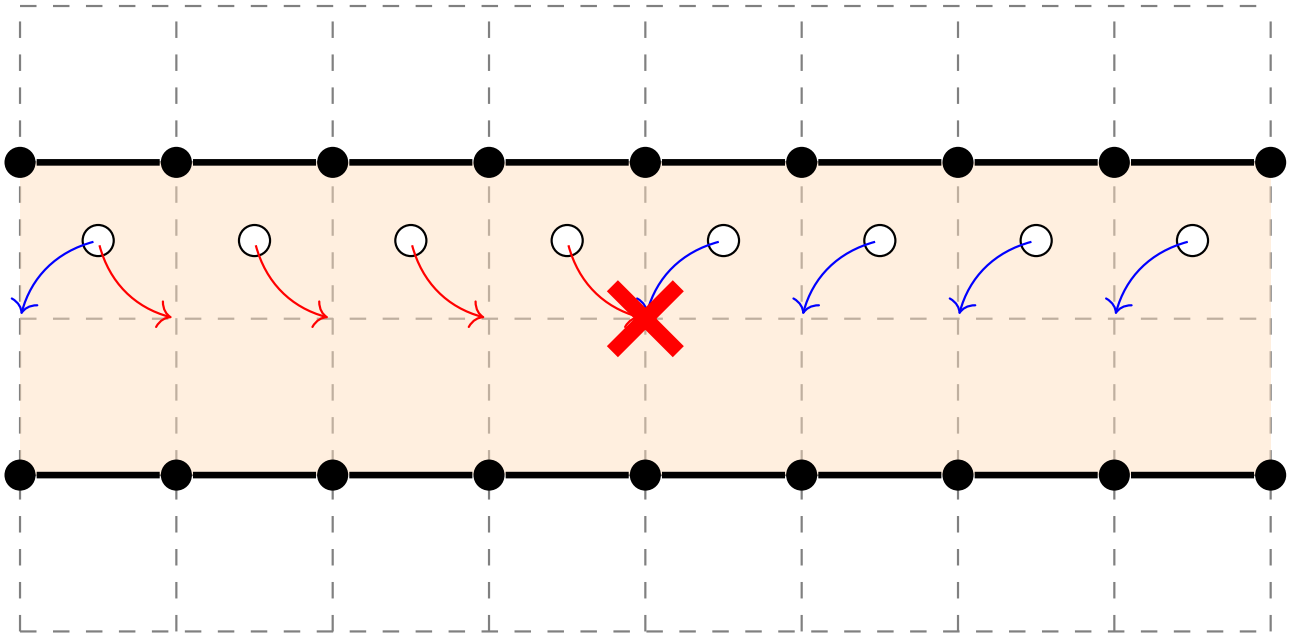
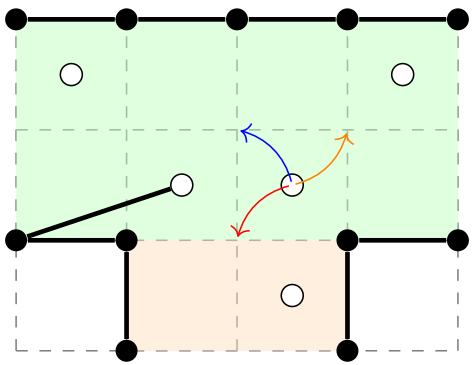
- Edges form tunnels that transmit pushes.
- Topological safety ensures consistency of transmission.



Clauses



- At the center, there is a decider vertex with (up to) three possibible target grid points.

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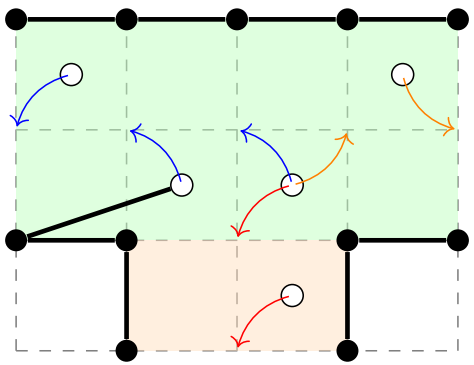

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- Following one arrow, rounding generates pushes.

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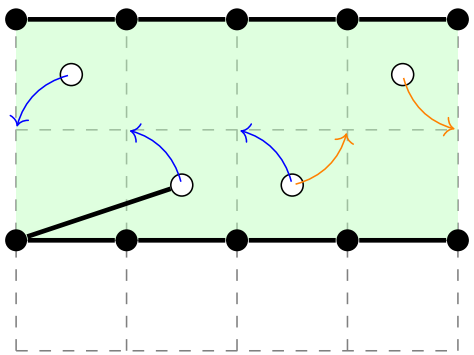

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- Blocking the bottom tunnel gives clause-gadgets for two variables.

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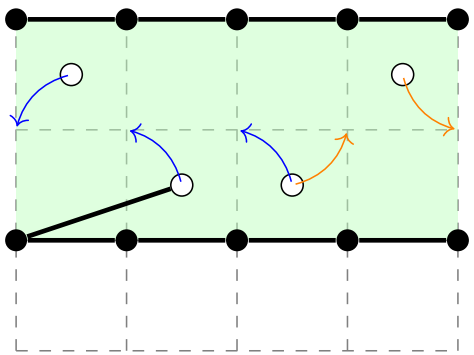

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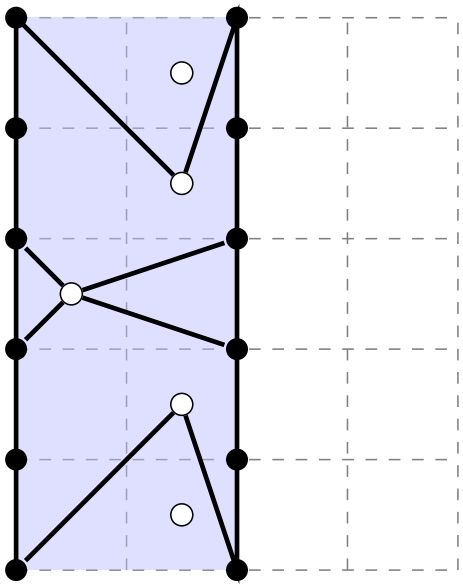
- Following one arrow, rounding generates pushes.
- Blocking the bottom tunnel gives clause-gadgets for two variables.
- All-unnegated gadgets are constructed mirroring at a horizontal line.

Variables



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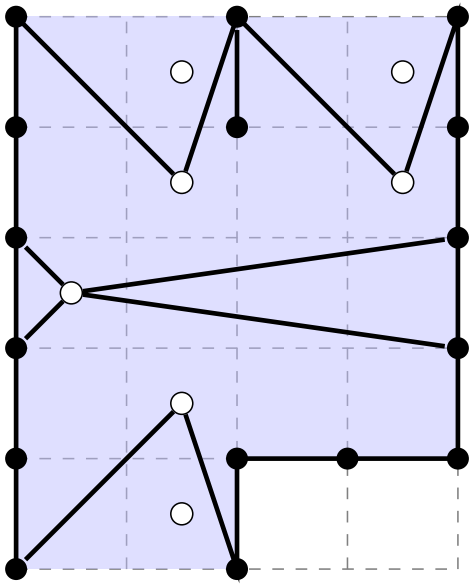

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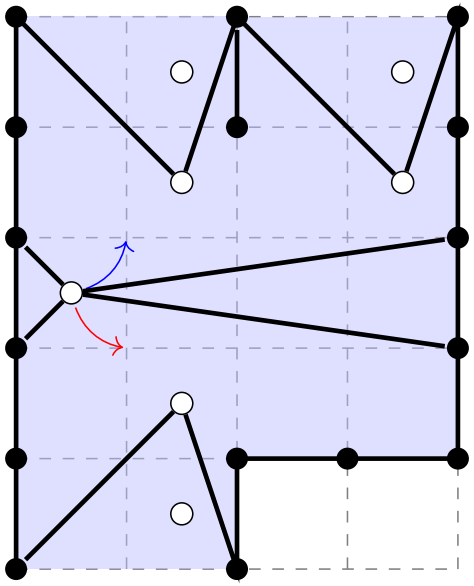

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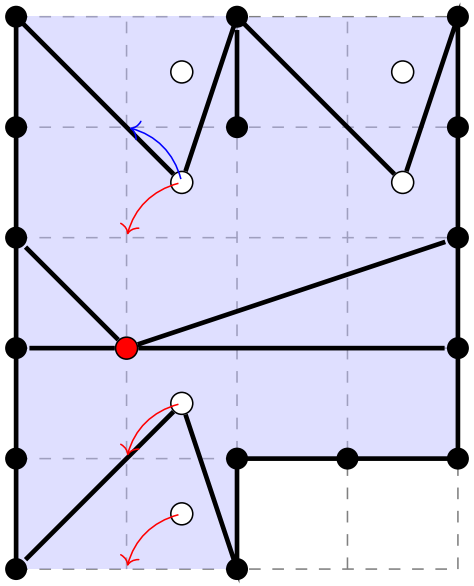

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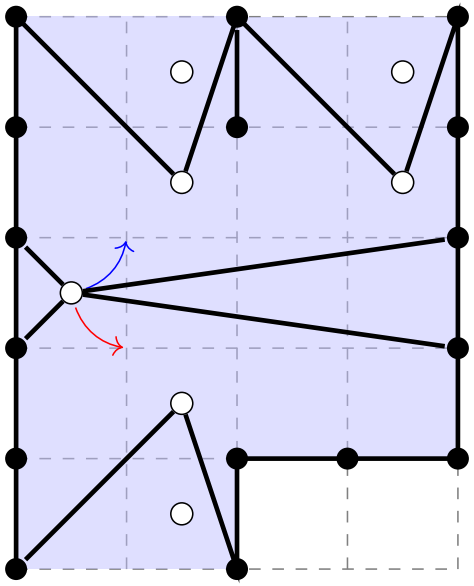

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- Grows horizontally with number of occurances.
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- Moving the assignment vertex up equals a TRUE-assignment, FALSE otherwise.

Hardness Proof

Theorem

Cost-bound Topologically Safe Snapping *is $\mathcal{NP}$-complete.*

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Euclidean Topologically Safe Snapping *with the objective to minimize* **maximum** *movement is $\mathcal{APX}$-hard.*

Integer Linear Program

Overview

Things to handle:

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- Unique vertex coordinates

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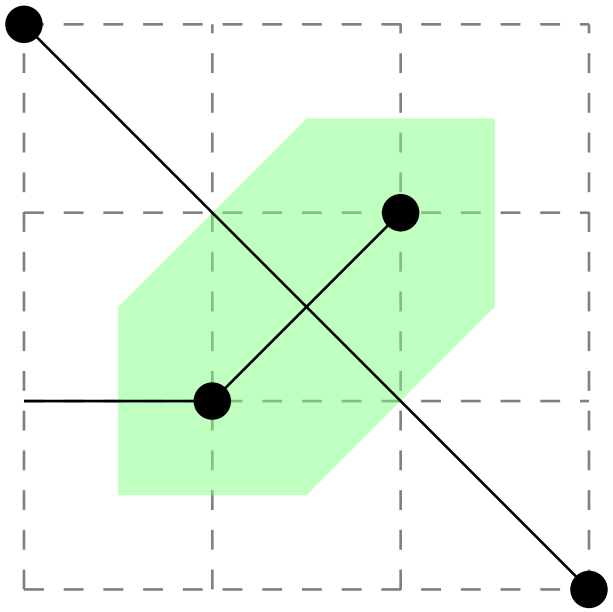

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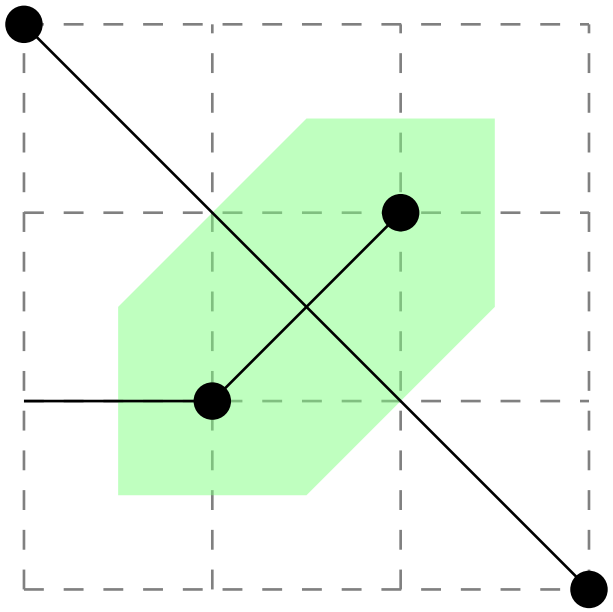

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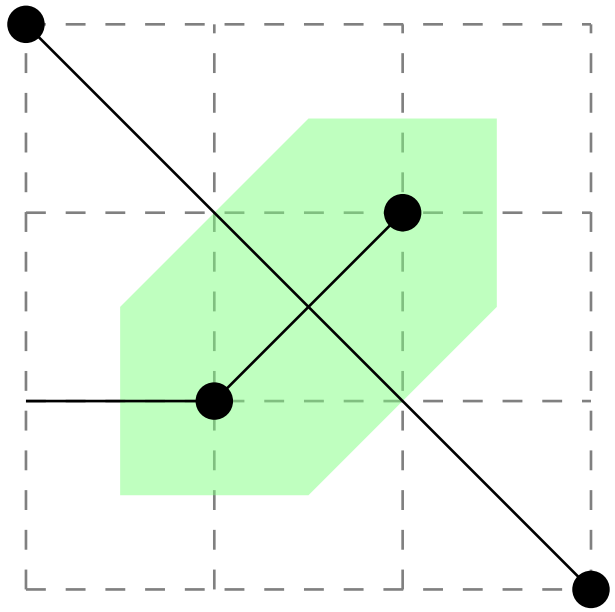

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- According to bounding box size:

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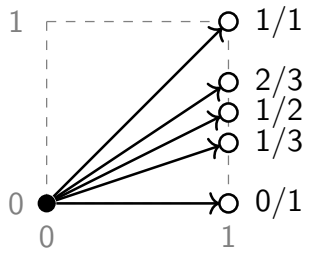

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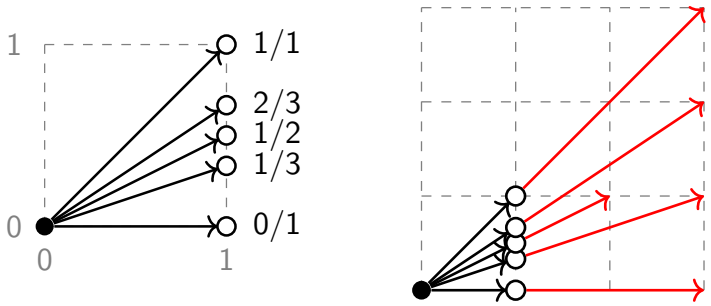

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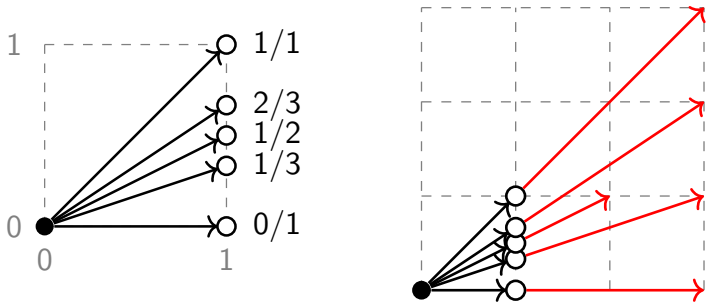

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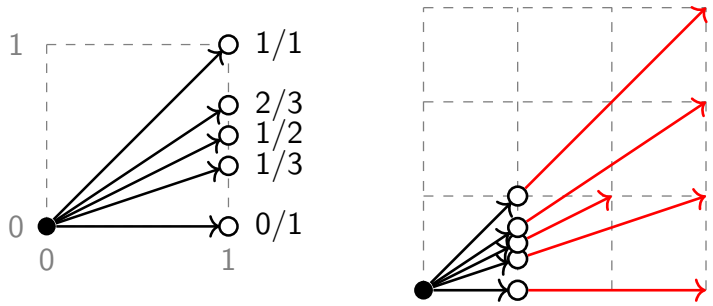

- Inside $[-k, k] \times [-k, k]$ area, there are $\Theta(k^2)$ directions to consider.
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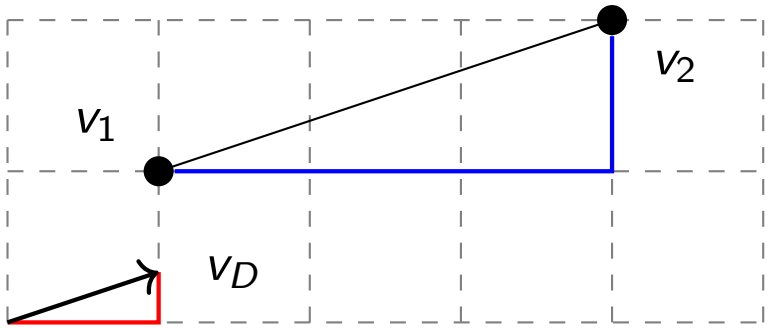

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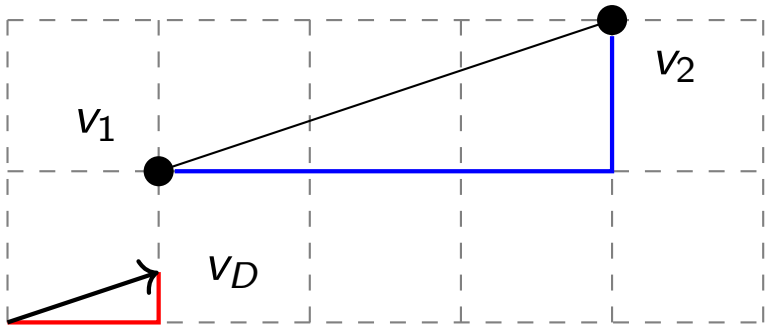

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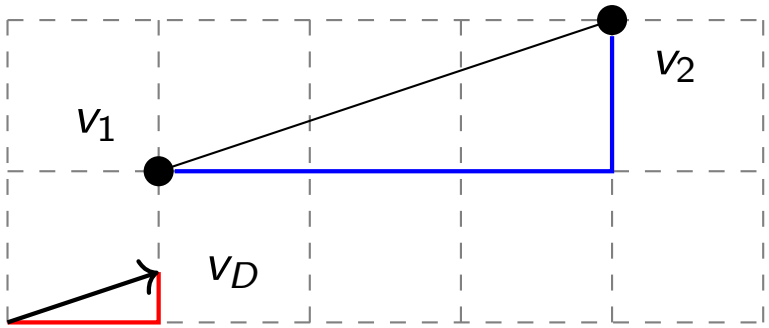

- Map edges to directions and compare the ordering of those directions to the given embedding.

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This ILP solves Topologically Safe Snapping.

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- We use delayed constraint generation to iteratively improve our model.
- We generate most constraints on demand: first iteration is simple rounding (with unique coordinates).

Experimental Evaluation

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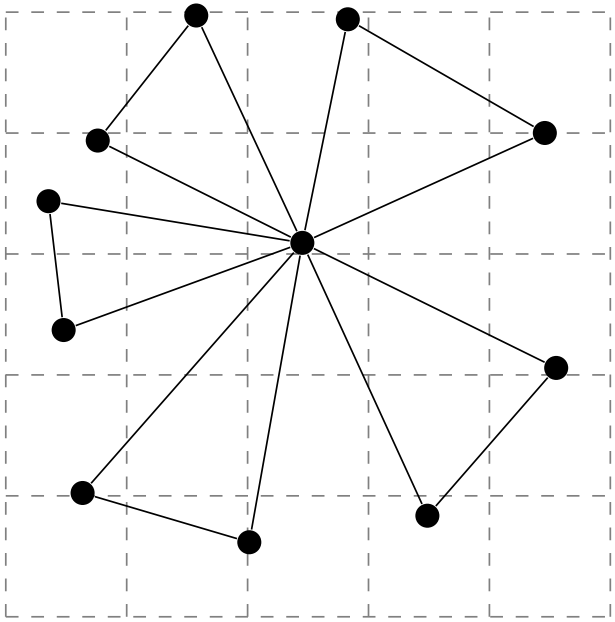
- Numbers of rows & columns before CPLEX presolving.
- Runtime in wall-clock time.
- For delayed constraint generation, time is accumulated total.

The Good



	Full	Delayed
rows		
cols		
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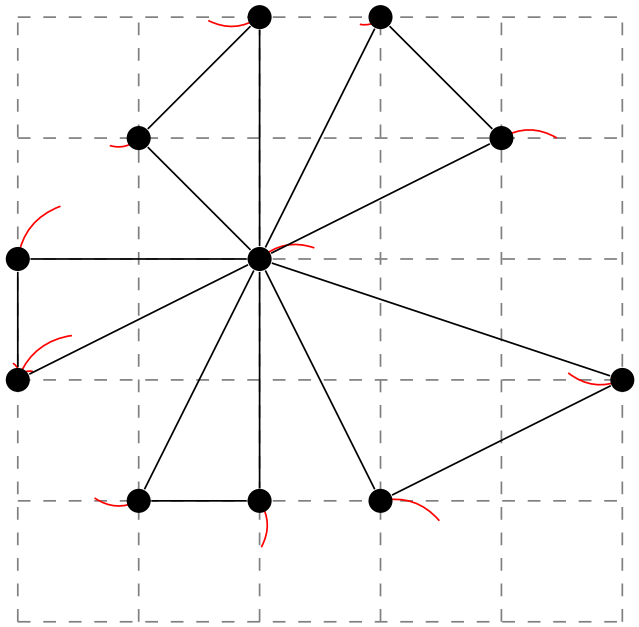

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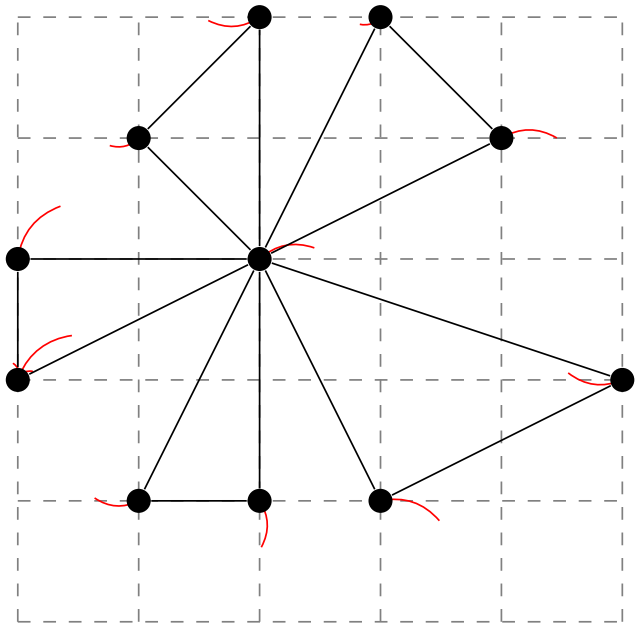

- Even small examples take several seconds to solve.

	Full	Delayed
rows	42 699	
cols	11 300	
time	10.6 s	

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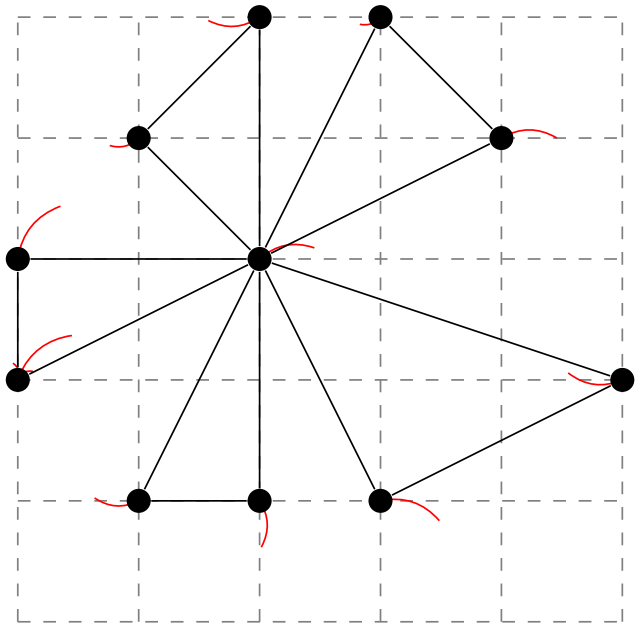

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	Full	Delayed
rows	42 699	
cols	11 300	
time	10.6 s	

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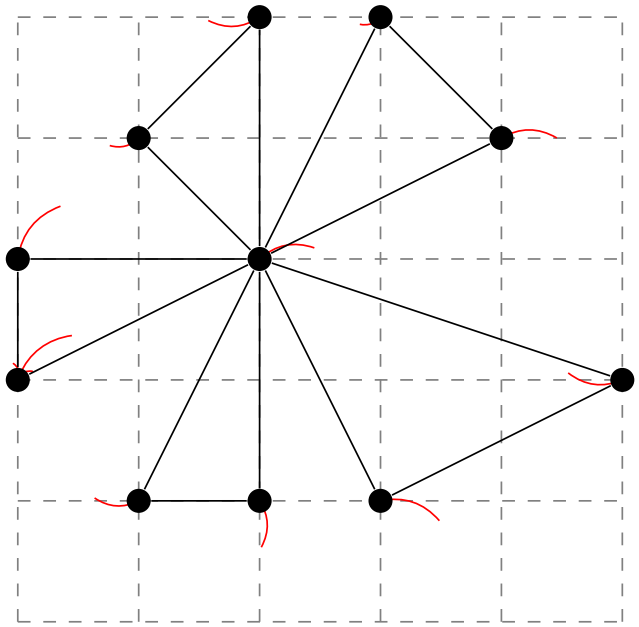

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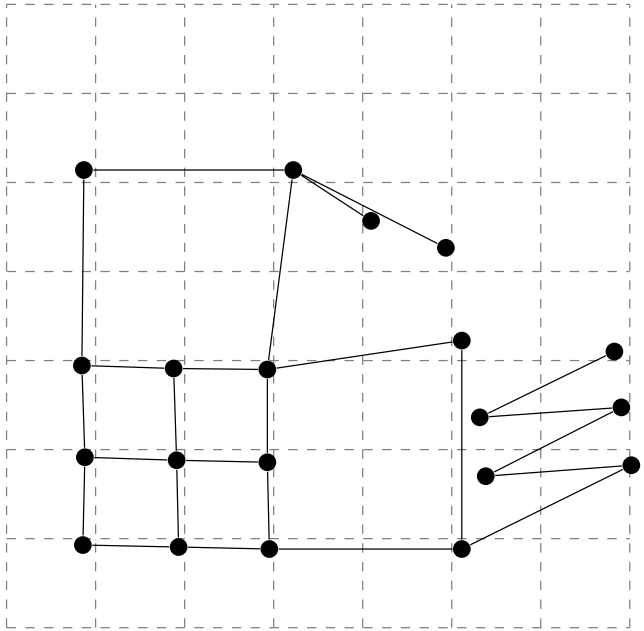
- Delayed constraint generation gives speed-up.

	Full	Delayed
rows	42 699	88
cols	11 300	110
time	10.6 s	0.5 s

The Bad



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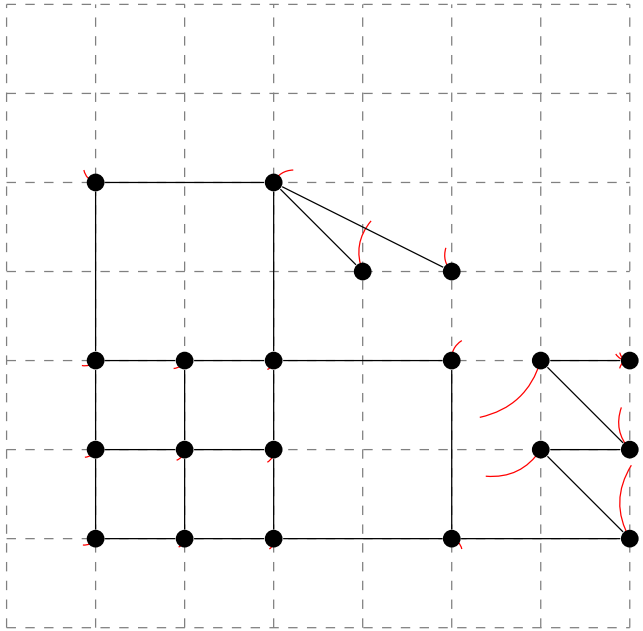

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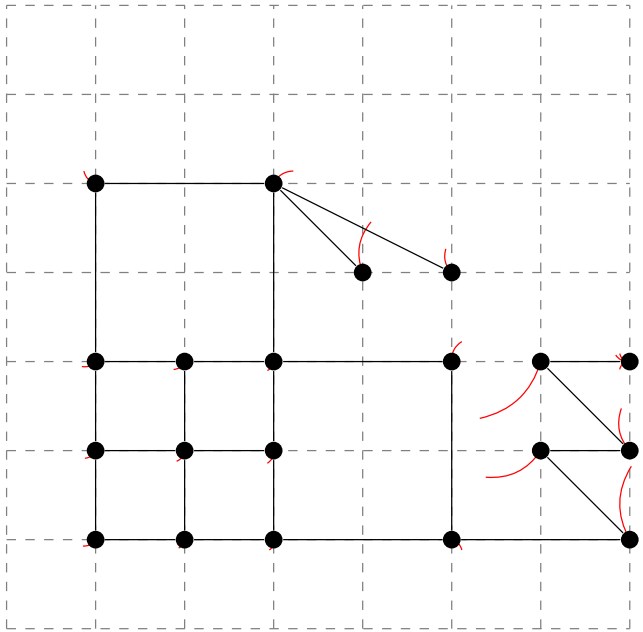

- We have canceled this computation after 10 minutes using the full model.

	Full	Delayed
rows	323 441	
cols	82 816	
time	†	

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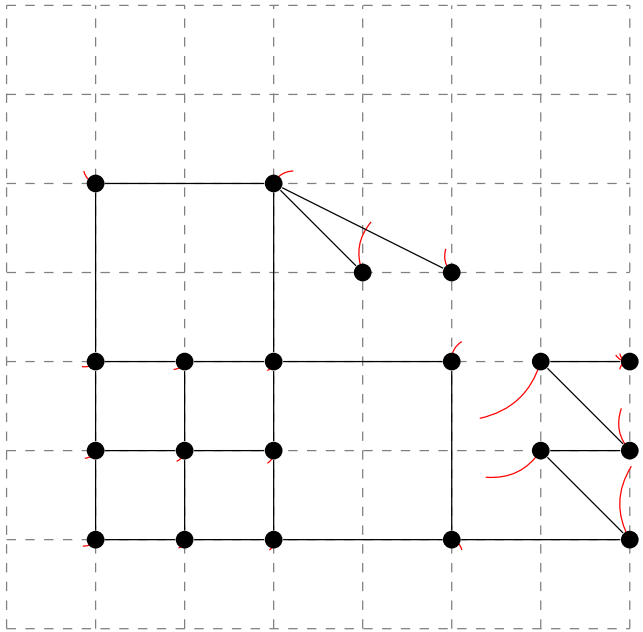

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rows	323 441	15 161
cols	82 816	4 044
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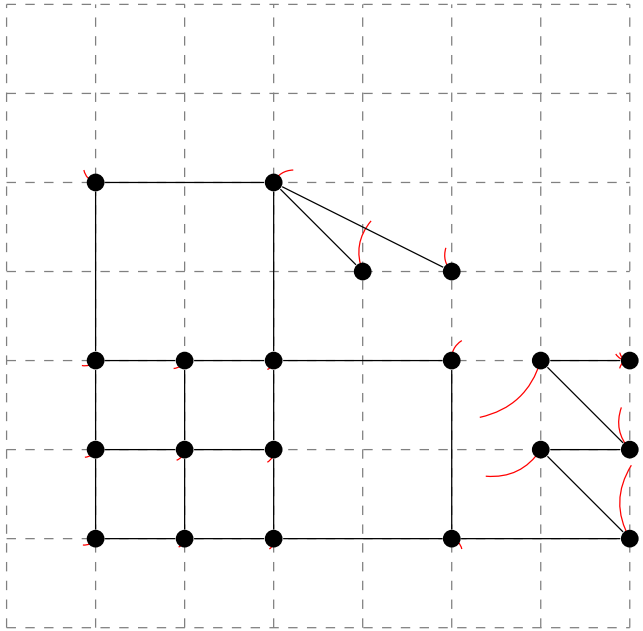

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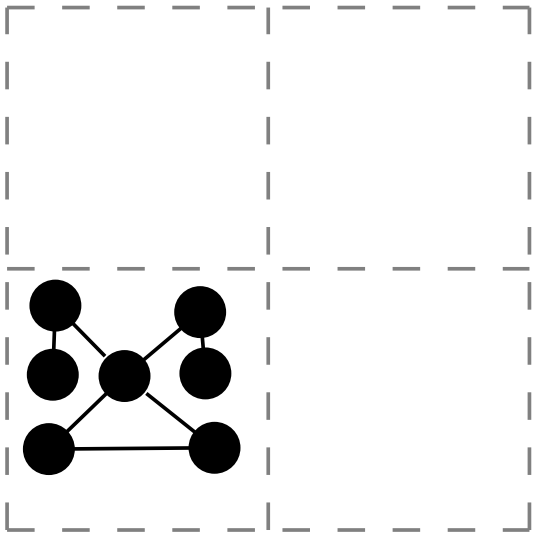
- ...waiting more than 3 minutes is too long for a graph on 20 vertices!

	Full	Delayed
rows	323 441	15 161
cols	82 816	4 044
time	†	211.6 s

The Ugly



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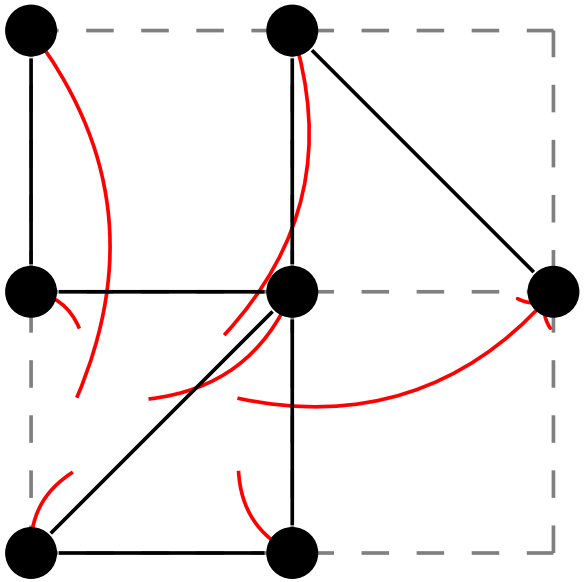

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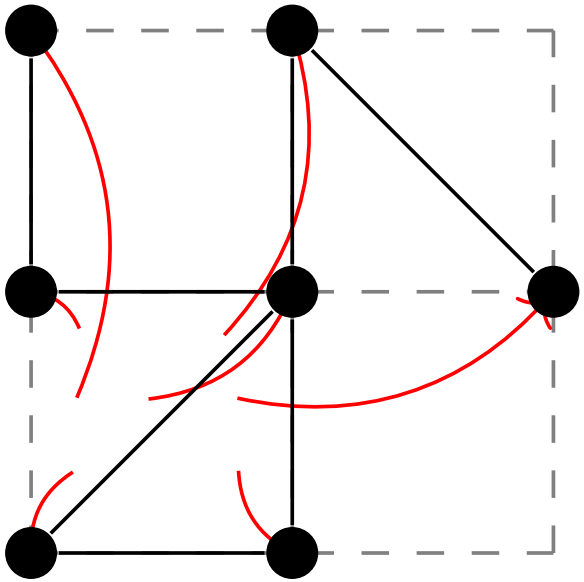

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rows	2 603	
cols	916	
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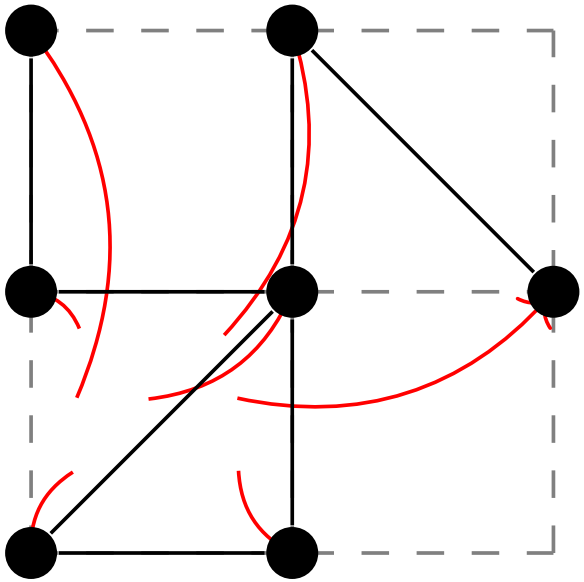

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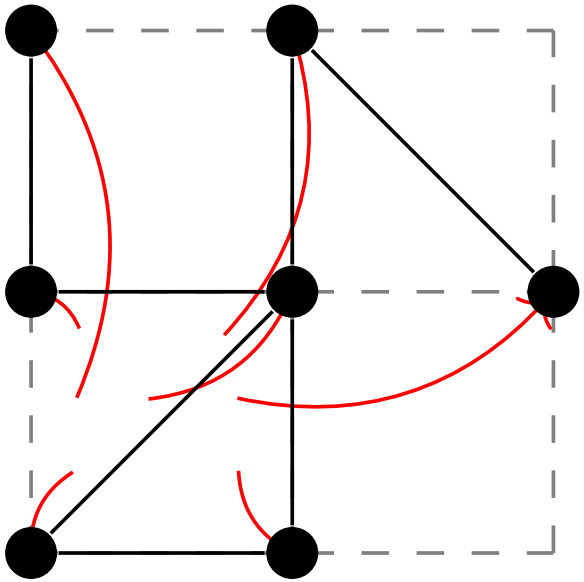

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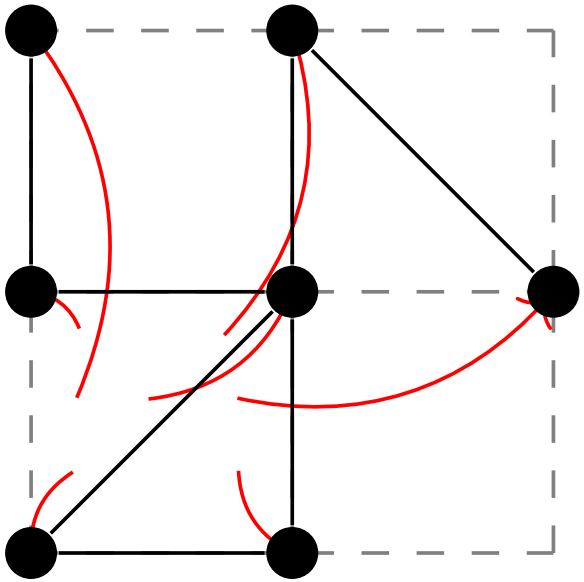

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- Questions about approximability remain open.