24TH INTERNATIONAL SYMPOSIUM ON GRAPH DRAWING AND NETWORK VISUALIZATION

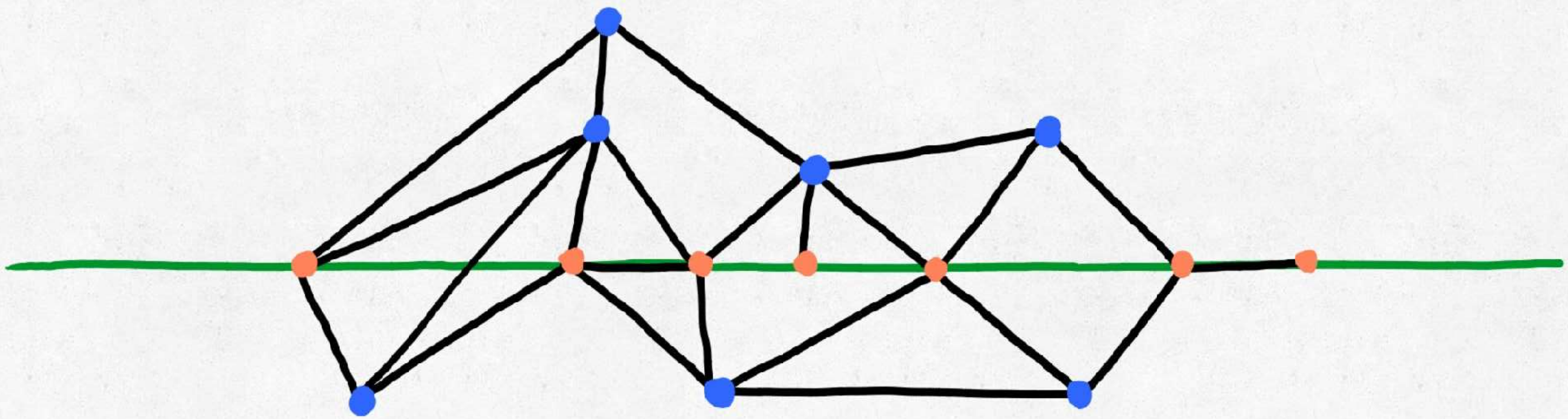
DRAWING PLANAR GRAPHS WITH MANY COLLINEAR VERTICES

GIORDANO DA LOZZO — ROMA TRE UNIVERSITY, ITALY
VIDA DUJMOVIĆ — UNIVERSITY OF OTTAWA, CANADA
FABRIZIO FRATI — ROMA TRE UNIVERSITY, ITALY
TAMARA MCHEDLIDZE — KIT, GERMANY
VINCENZO ROSELLI — ROMA TRE UNIVERSITY, ITALY

THE GOAL

CONSTRUCT PLANAR STRAIGHT-LINE DRAWINGS

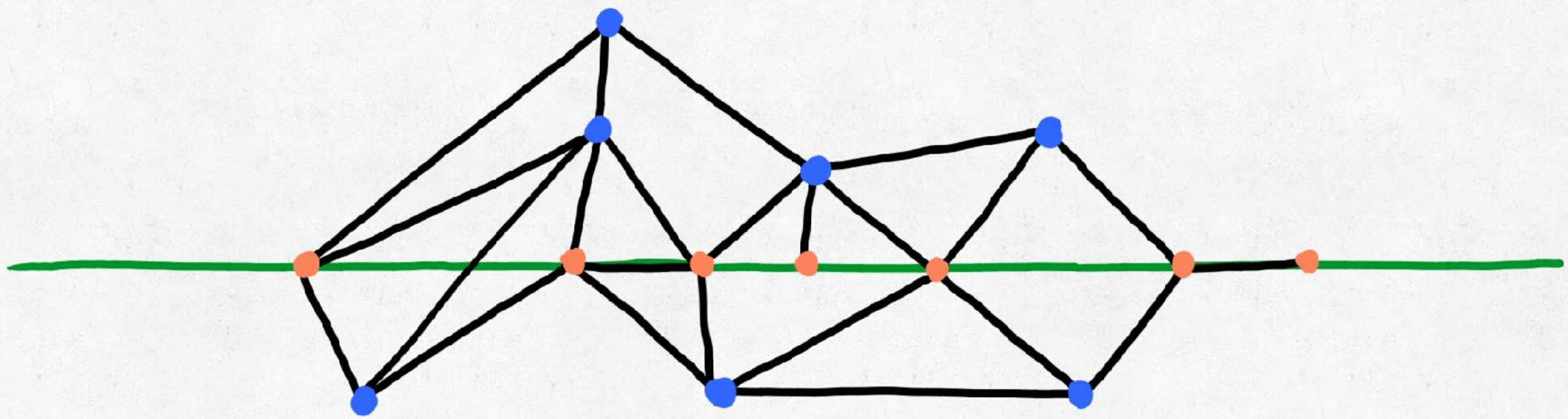
WITH MANY COLLINEAR VERTICES!



$$f(n) = \min_{n\text{-VERTEX PLANAR GRAPH } G} \left\{ \max_{\text{STRAIGHT-LINE PLANAR DRAWING } \Gamma \text{ OF } G} \left\{ \begin{array}{l}\text{MAXIMUM NUMBER OF} \\ \text{COLLINEAR VERTICES IN } \Gamma\end{array} \right\}\right\}$$

COLLINEAR SETS

GIVEN A PLANAR GRAPH $G = (V, E)$, A SET $S \subseteq V$ IS A COLLINEAR SET IF THERE EXISTS A PLANAR STRAIGHT-LINE DRAWING OF G IN WHICH THE VERTICES IN S ARE COLLINEAR.



$$f(n) = \min_{n\text{-VERTEX PLANAR GRAPH } G} \left\{ \max_{\text{COLLINEAR SET } S \text{ IN } G} \{ |S| \} \right\}$$

STATE OF THE ART

THE PROBLEM WAS INTRODUCED IN [RAVSKY, VERBITSKY 2011]

PLANAR GRAPHS	$f(n) \in \Omega(\sqrt{n})$ [DUJMOVIĆ 2015]	$f(n) \in O(n^{0.986})$ [RAVSKY, VERBITSKY 2011]
OUTERPLANAR GRAPHS	$f(n) \geq (n+1)/2$ [GOAOC et al. 2009 ; RAVSKY, VERBITSKY 2011]	
GRAPHS WITH TREEWIDTH AT MOST 2	$f(n) \geq n/30$ [RAVSKY, VERBITSKY 2011]	

OUR RESULTS

THEOREM 1 EVERY n-VERTEX PLANAR GRAPH OF TREEWIDTH AT MOST THREE HAS A COLLINEAR SET WITH SIZE $\left\lceil \frac{n-3}{8} \right\rceil$.

THEOREM 2 EVERY n-VERTEX TRICONNECTED CUBIC PLANAR GRAPH HAS A COLLINEAR SET WITH SIZE $\left\lceil \frac{n}{4} \right\rceil$.

THEOREM 3 EVERY PLANAR GRAPH WITH TREEWIDTH k HAS A COLLINEAR SET WITH SIZE $\Omega(k^2)$.

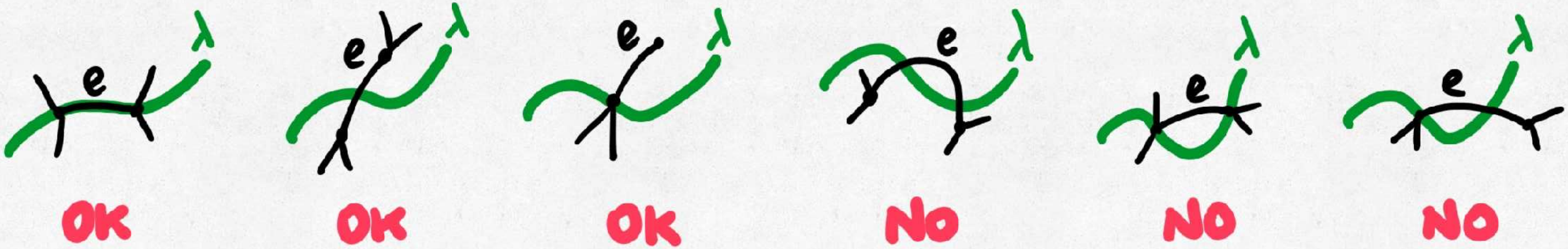
OUR TOOL:

A CHARACTERIZATION USING GOOD CURVES

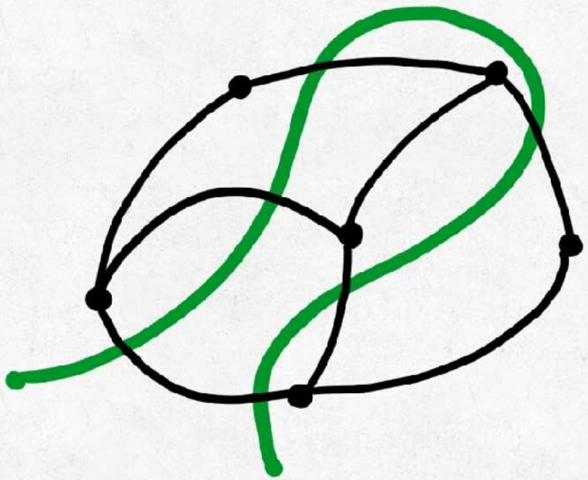
GOOD CURVES

CONSIDER A PLANAR DRAWING Γ OF A PLANAR GRAPH G.
AN OPEN SIMPLE CURVE λ IS GOOD IF:

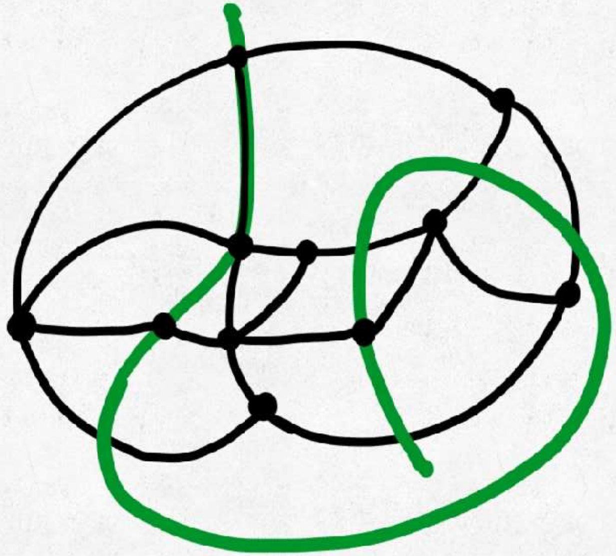
1) FOR EVERY EDGE $e \in E(G)$, EITHER λ CONTAINS e ENTIRELY, OR λ INTERSECTS e AT MOST ONCE (THE END-VERTICES OF e IN λ COUNT).



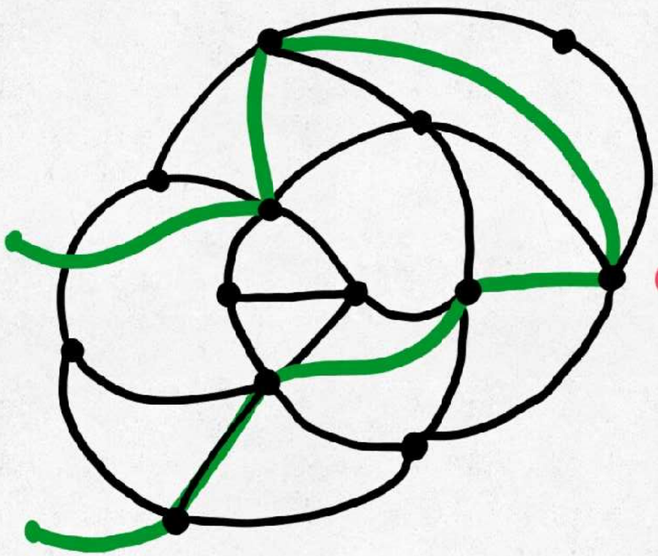
2) BOTH THE END-VERTICES OF λ ARE INCIDENT TO THE UNBOUNDED REGION OF THE PLANE DEFINED BY Γ AND λ.



NOT GOOD



NOT GOOD

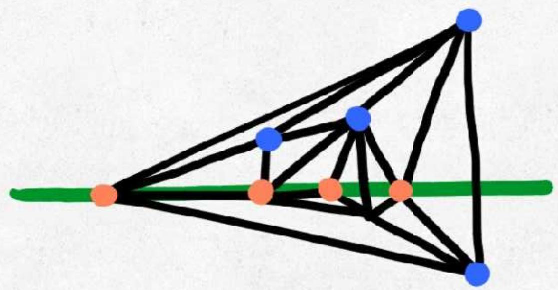


GOOD

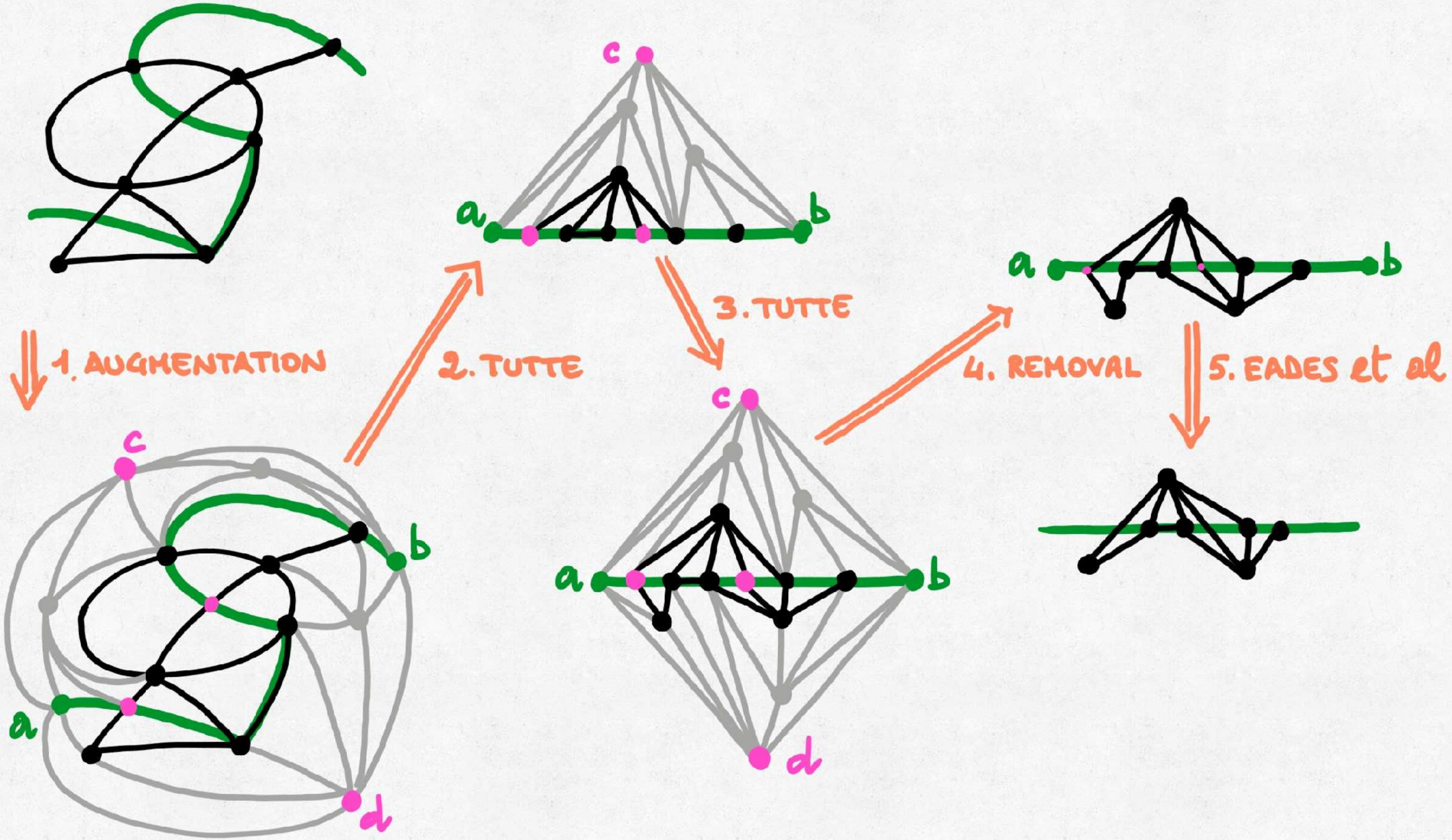
CHARACTERIZATION

A PLANAR GRAPH HAS A COLLINEAR SET WITH X VERTICES IF AND ONLY IF IT ADMITS A GOOD CURVE PASSING THROUGH X OF ITS VERTICES.

PROOF: THE NECESSITY IS TRIVIAL

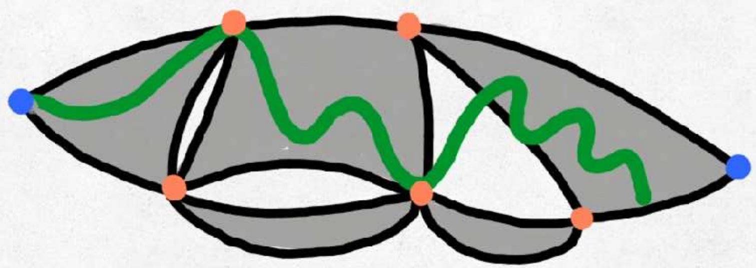


FOR THE SUFFICIENCY...


1. AUGMENTATION
2. TUTTE
3. TUTTE
4. REMOVAL
5. EADES et al.
a
b
c
d

THEOREM 2 EVERY n-VERTEX TRICONNECTED CUBIC PLANAR GRAPH HAS A COLLINEAR SET WITH SIZE $\left\lceil \frac{n}{4} \right\rceil$.

- BY INDUCTION ON A FAMILY OF GRAPHS, CALLED STRONG CIRCUIT GRAPHS, WIDER THAN TRICONNECTED CUBIC PLANAR GRAPHS

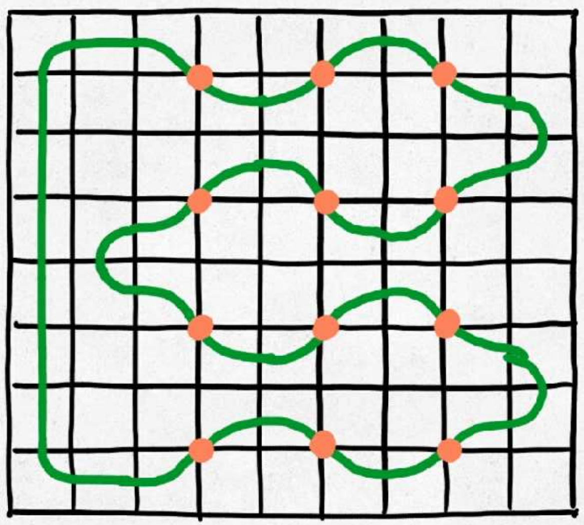


- USES A STRUCTURAL DECOMPOSITION FOR STRONG CIRCUIT GRAPHS DERIVED FROM A PAPER BY CHEN AND YU [CHEN, YU 2012]

THEOREM 3 EVERY PLANAR GRAPH WITH TREEWIDTH k HAS A COLLINEAR SET WITH SIZE $\Omega(k^2)$.

- USES A RESULT OF ROBERTSON *et al.* [ROBERTSON, SEYMOUR, THOMAS 1994]:

  IF G IS A PLANAR GRAPH WITH TREEWIDTH k, THEN G CONTAINS A $\Omega(k) \times \Omega(k)$ - SIZE GRID AS A MINOR



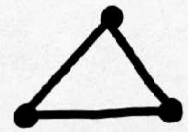
THEOREM 1 EVERY n-VERTEX PLANAR GRAPH OF TREEWIDTH AT MOST THREE HAS A COLLINEAR SET WITH SIZE $\left\lceil \frac{n-3}{8} \right\rceil$.

- EVERY PLANAR GRAPH OF TREEWIDTH AT MOST THREE CAN BE AUGMENTED WITH DUMMY EDGES TO A PLANE 3-TREE [KRATOCHVÍL, VANER 2012]

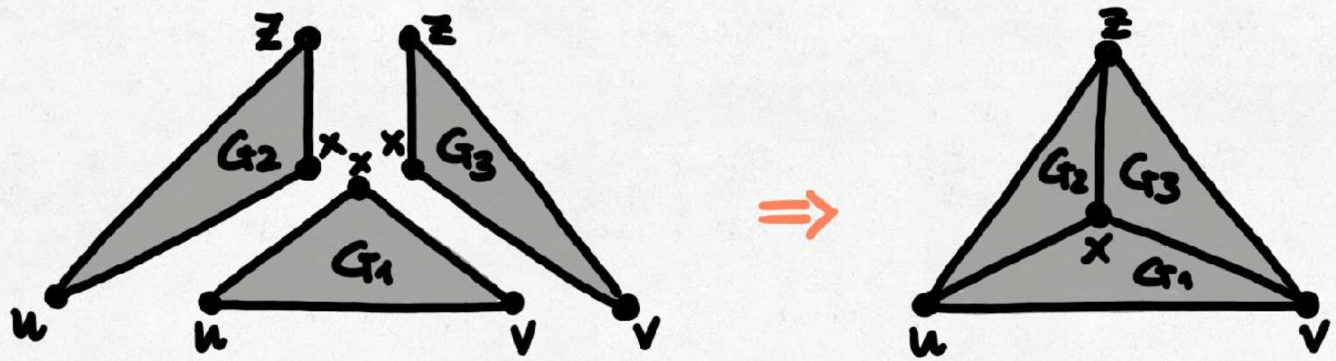
  $\Rightarrow$ IT SUFFICES TO PROVE THE THEOREM FOR PLANE 3-TREES

PLANE 3-TREES

- A PLANE 3-CYCLE IS A PLANE 3-TREE (THE EMPTY PLANE 3-TREE)



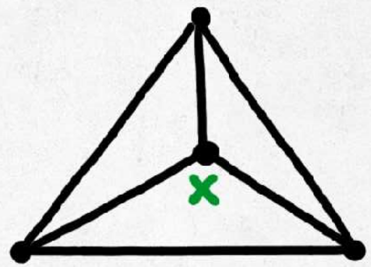
- CONSIDER A PLANE K_4 AND PLUG THREE PLANE 3-TREES INSIDE ITS INTERNAL FACES. THE RESULTING PLANE GRAPH IS A PLANE 3-TREE G



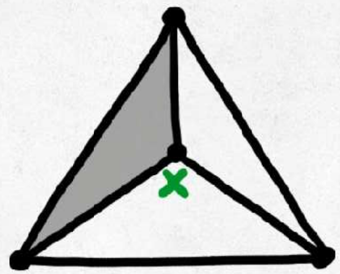
 - G_1, G_2, AND G_3 ARE THE CHILDREN OF G
 - x IS THE CENTRAL VERTEX OF G

VERTEX TYPES

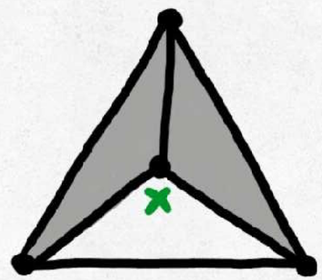
AN INTERNAL VERTEX x OF G IS OF ONE OF THE FOLLOWING TYPES:



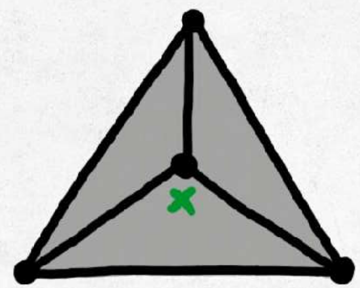
TYPE A



TYPE B



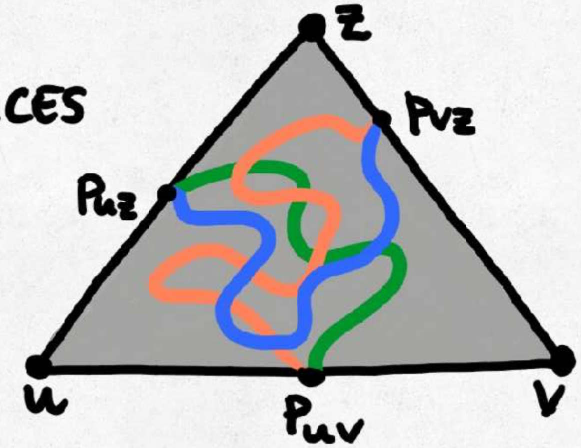
TYPE C



TYPE D

GENERAL IDEA

- CONSTRUCT THREE GOOD CURVES λ_u, λ_v, AND λ_z CONNECTING INTERNAL POINTS OF EDGES uv AND uz, uv AND vz, AND uz AND vz, RESPECTIVELY.



- λ_u, λ_v, AND λ_z MIGHT PASS THROUGH THE SAME VERTICES OF G, AND MIGHT CROSS EACH OTHER ARBITRARILY.

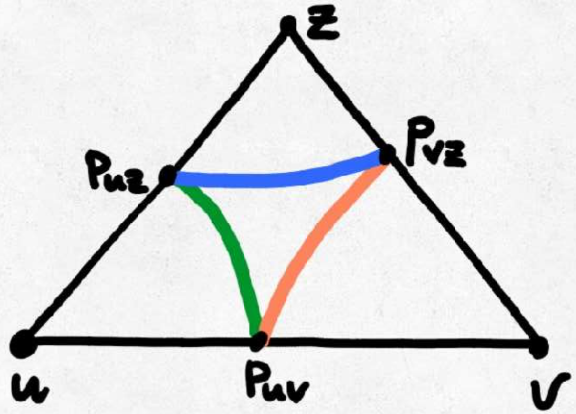
- EACH OF λ_u, λ_v, AND λ_z PASSES THROUGH ALL THE VERTICES OF TYPE A AND THROUGH NO VERTEX OF TYPE C OR D. ALTOGETHER λ_u, λ_v, AND λ_z PASS THROUGH "MANY" VERTICES OF TYPE B.

- THE TOTAL NUMBER OF VERTICES THE CURVES PASS THROUGH IS AT LEAST $\frac{3n-9}{8}$, FROM WHICH THE THEOREM FOLLOWS.

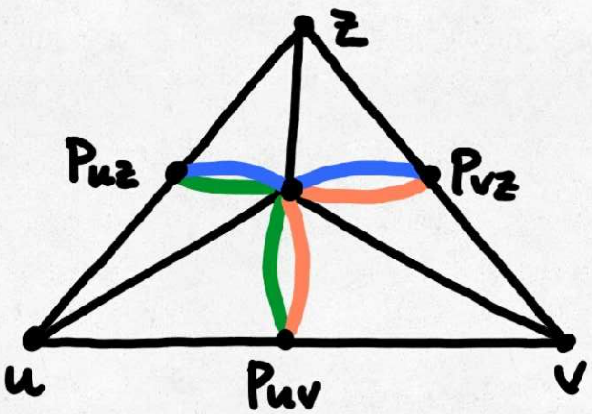
CONSTRUCTION - BASE CASES

LET m BE THE NUMBER OF INTERNAL VERTICES OF G (I.E., $m = n-3$).

$m=0$



$m=1$

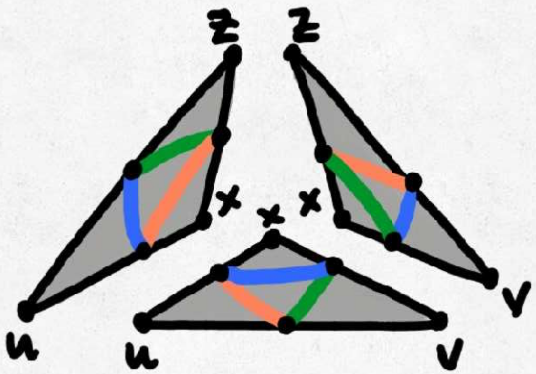


CONSTRUCTION - INDUCTIVE CASE - X OF TYPE C OR D

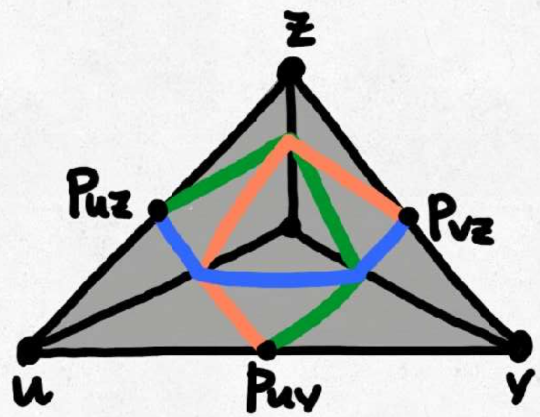
$m > 1$ AND THE CENTRAL VERTEX x OF G IS OF TYPE C OR D

$\Rightarrow$ INDUCTIVELY CONSTRUCT CURVES FOR THE CHILDREN OF G.

JOIN THEM IN ORDER TO OBTAIN λ_u, λ_v, AND λ_z



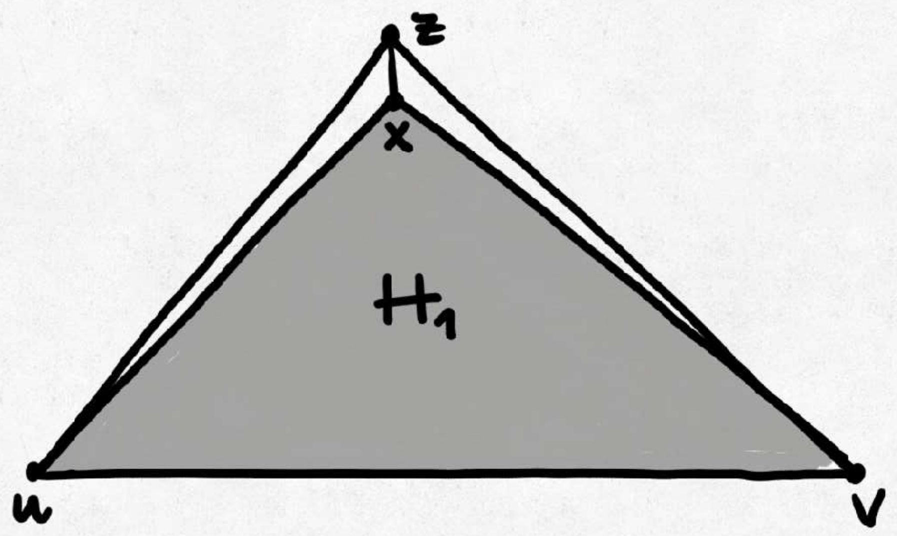
$\Longrightarrow$



CONSTRUCTION - INDUCTIVE CASE - X OF TYPE B

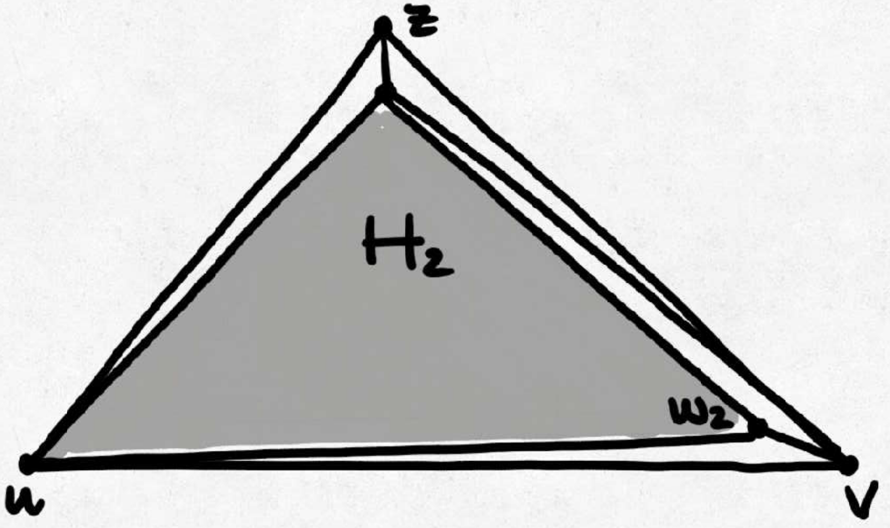
$m > 1$ AND THE CENTRAL VERTEX x OF G IS OF TYPE B

$\Rightarrow$ WE NEED TO FURTHER EXPLORE THE STRUCTURE OF G, IN ORDER TO RECOVER A MAXIMAL SEQUENCE OF VERTICES OF TYPE B : A B-CHAIN



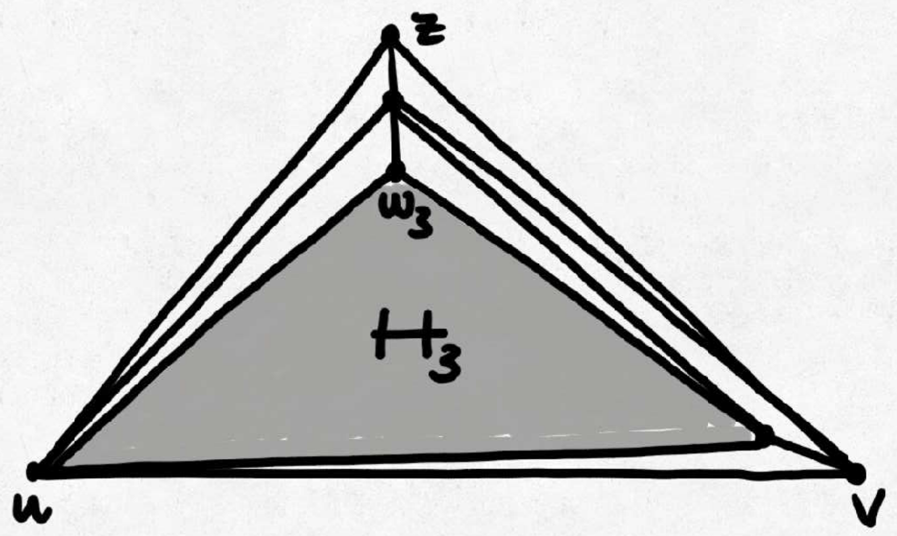
B-CHAIN : $x = w_1$

CONSTRUCTION - INDUCTIVE CASE - X OF TYPE B



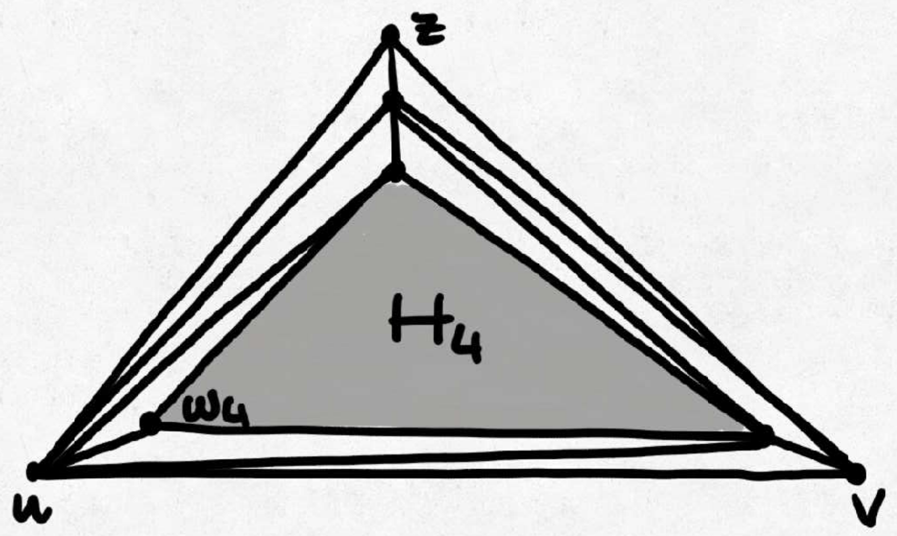
B-CHAIN : $x = w_1, w_2$

CONSTRUCTION - INDUCTIVE CASE - X OF TYPE B



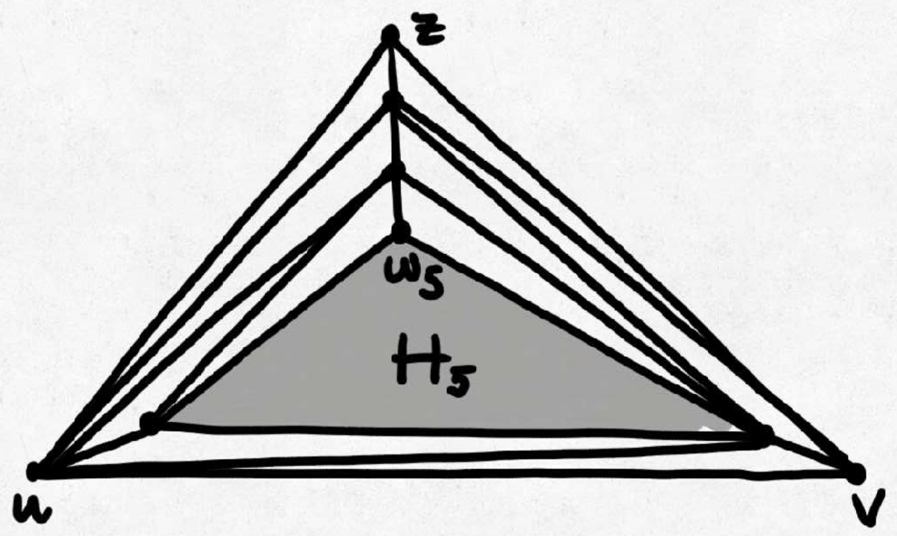
B-CHAIN : $x = w_1, w_2, w_3$

CONSTRUCTION - INDUCTIVE CASE - X OF TYPE B



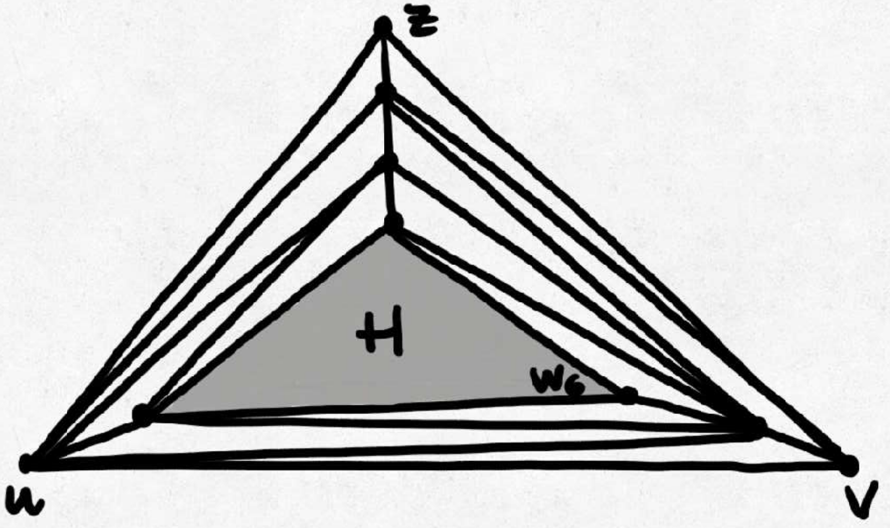
B-CHAIN : $x = w_1, w_2, w_3, w_4$

CONSTRUCTION - INDUCTIVE CASE - X OF TYPE B



B-CHAIN : $x = w_1, w_2, w_3, w_4, w_5$

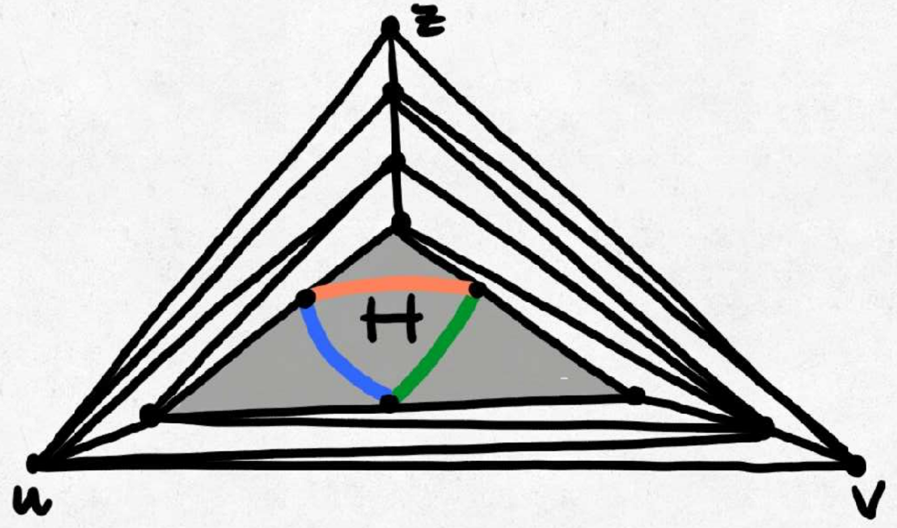
CONSTRUCTION - INDUCTIVE CASE - X OF TYPE B



B-CHAIN : $x = w_1, w_2, w_3, w_4, w_5, w_6$

CONSTRUCTION - INDUCTIVE CASE - X OF TYPE B

$\Rightarrow$ INDUCTIVELY CONSTRUCT CURVES FOR H

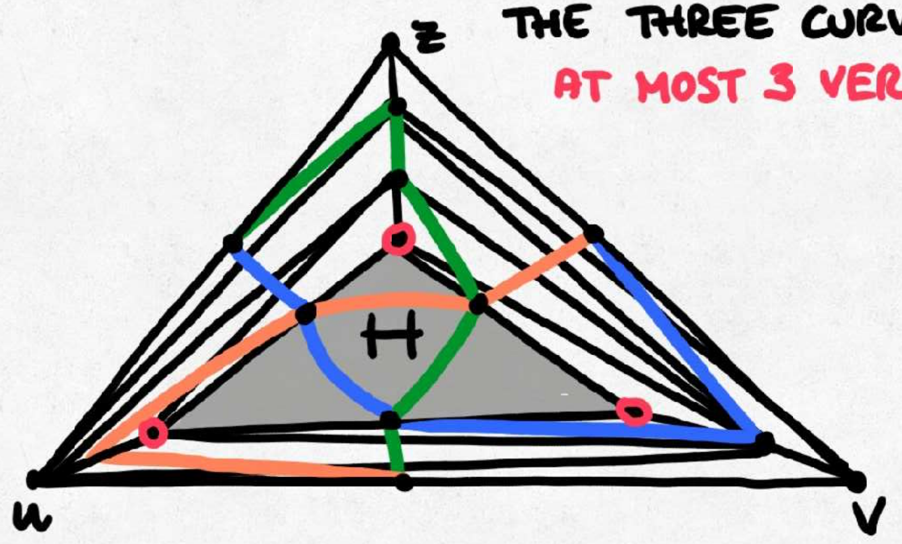


B-CHAIN $\boxed{x = w_1, w_2, w_3, w_4, w_5, w_6}$

CONSTRUCTION - INDUCTIVE CASE - X OF TYPE B

$\Rightarrow$ COMPLETE THE CURVES FOR G BY PASSING THROUGH THE VERTICES OF THE B-CHAIN

THE THREE CURVES λ_u, λ_v, AND λ_z MISS AT MOST 3 VERTICES OF THE B-CHAIN



B-CHAIN $\boxed{x = w_1, w_2, w_3, w_4, w_5, w_6}$

SOME COUNTING

(1) $a(G) + b(G) + c(G) + d(G) = m$

(2) $s(G) \geq 3a(G)$

$\Rightarrow$ IF $a(G) \in \Omega(m)$ **WIN**

(3) $a(G) = c(G) + 2d(G) + 1$

$\Rightarrow$ IF $c(G)$ OR $d(G) \in \Omega(m)$ **WIN**

OTHERWISE $a(G), c(G), d(G) \in o(m)$

AND $b(G) \in \Omega(m)$

(4) $s(G) \geq b(G) - x(G)$

(5) $x(G) \leq 3h(G)$

(6) $h(G) \leq 2c(G) + 3d(G) + 1$

$\Rightarrow h(G) \in o(m)$ **WIN**

$a(G)$ = # TYPE-A VERTICES

$b(G)$ = # TYPE-B VERTICES

$c(G)$ = # TYPE-C VERTICES

$d(G)$ = # TYPE-D VERTICES

m = # INTERNAL VERTICES

$s(G)$ = TOTAL # VERTICES THE CURVES PASS THROUGH

$h(G)$ = # OF B-CHAINS

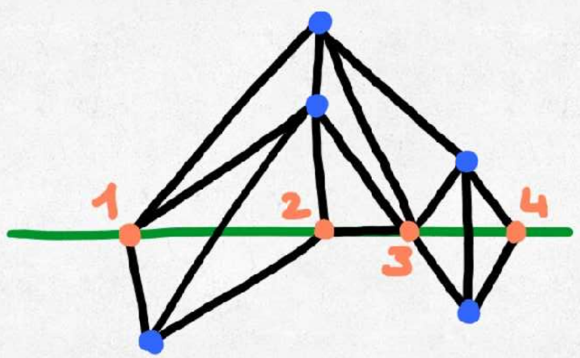
$x(G)$ = # TYPE-B VERTICES NONE OF THE CURVES PASSES THROUGH

THEOREM 1 EVERY n-VERTEX PLANAR GRAPH OF TREEWIDTH AT MOST THREE HAS A COLLINEAR SET WITH SIZE $\left\lceil \frac{n-3}{8} \right\rceil$.

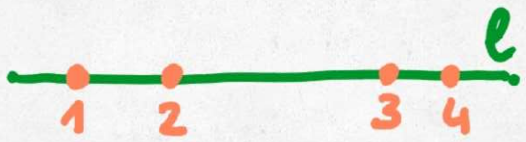
THEOREM 1' EVERY n-VERTEX PLANAR GRAPH OF TREEWIDTH AT MOST THREE HAS A FREE COLLINEAR SET WITH SIZE $\left\lceil \frac{n-3}{8} \right\rceil$.

FREE COLLINEAR SETS

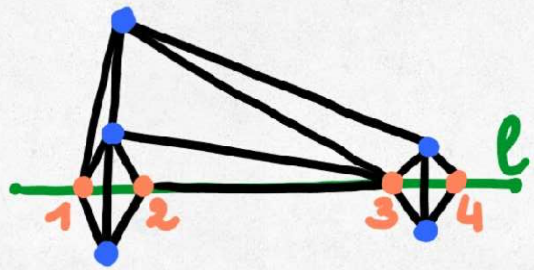
FOR A PLANAR GRAPH $G=(V,E)$, A COLLINEAR SET $S \subseteq V$ IS **FREE** IF THERE EXISTS A TOTAL ORDER σ OF S SUCH THAT, FOR ANY SET P OF $|S|$ POINTS ON A STRAIGHT LINE ℓ, GRAPH G HAS A PLANAR STRAIGHT-LINE DRAWING ψ IN WHICH THE VERTICES IN S ARE MAPPED TO THE POINTS IN P AND THEIR ORDER ALONG ℓ IS σ.



σ: 1, 2, 3, 4



P



ψ

STATE OF THE ART

THE PROBLEM WAS INTRODUCED IN [RAVSKY, VERBITSKY 2011]

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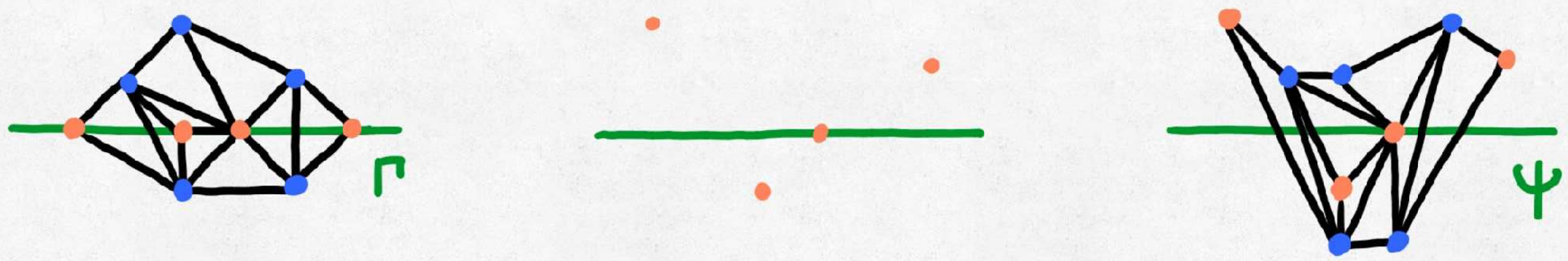
OUTERPLANAR GRAPHS	$g(n) \geq (n+1)/2$ [GOAOC et al. 2009 ; RAVSKY, VERBITSKY 2011]	
GRAPHS WITH TREEWIDTH AT MOST 2	$g(n) \geq n/30$ [RAVSKY, VERBITSKY 2011]	
GRAPHS WITH TREEWIDTH AT MOST 3	$g(n) \geq \lceil \frac{n-3}{8} \rceil$ [THIS PAPER]	

APPLICATIONS TO OTHER GRAPH DRAWING PROBLEMS

POSSIBLE DUE TO THE FOLLOWING LEMMA [BOSE, DUJMOVIĆ, HURTADO, LANGERMAN, MORIN, WOOD 2009]

LEMMA LET $G=(V,E)$ BE A PLANAR GRAPH THAT HAS A PLANAR STRAIGHT-LINE DRAWING Γ IN WHICH A SET S OF VERTICES LIE ON THE x-AXIS.
FOR ANY ASSIGNMENT OF y-COORDINATES TO THE VERTICES IN S, THERE EXISTS A PLANAR STRAIGHT-LINE DRAWING Ψ OF G IN WHICH EVERY VERTEX IN S HAS THE SAME x-COORDINATE AS IN Γ AND HAS THE ASSIGNED y-COORDINATE.



UNTANGLING

GIVEN A STRAIGHT-LINE DRAWING Γ OF A PLANAR GRAPH G, THE GOAL IS TO CONSTRUCT A PLANAR STRAIGHT-LINE DRAWING Ψ OF G SO THAT AS MANY VERTICES AS POSSIBLE HAVE THE SAME LOCATION IN Ψ AND IN Γ.

KNOWN BOUNDS

PLANAR GRAPHS : $\Omega(n^{0.25})$ [BOSE et al. 2009] $O(n^{0.495})$ [CANO et al. 2014]

TREES, OUTERPLANAR GRAPHS, PLANAR GRAPHS OF TREEWIDTH 2 : $\theta(\sqrt{n})$ [BOSE et al. 2009; GOAOC et al. 2009; RAVSKY AND VERBITSKY 2011]

PLANAR GRAPHS OF TREEWIDTH AT MOST THREE : $\theta(\sqrt{n})$ [THIS PAPER]

UNIVERSAL POINT SUBSETS

A SET P OF $k \leq n$ POINTS SUCH THAT EVERY n-VERTEX PLANAR GRAPH ADMITS A PLANAR STRAIGHT-LINE DRAWING IN WHICH k OF ITS VERTICES ARE MAPPED TO THE POINTS IN P.

KNOWN BOUNDS

PLANAR GRAPHS : $\Omega(\sqrt{n})$ [ANGELINI et al. 2012 ; DUJMOVIĆ 2015]

OUTERPLANAR GRAPHS : n [GRITZMANN et al. 1991; CASTAÑEDA et al. 1996; BOSE 2002]

PLANAR GRAPHS OF TREEWIDTH AT MOST THREE : $\Theta(n)$ [THIS PAPER]

COLUMN PLANAR SETS

FOR A PLANAR GRAPH $G=(V,E)$ A COLUMN PLANAR SET IS A SET $S \subseteq V$ WITH THE FOLLOWING PROPERTY: THERE EXISTS AN x-COORDINATE ASSIGNMENT TO THE VERTICES IN S SUCH THAT, FOR ANY y-COORDINATE ASSIGNMENT TO THE VERTICES IN S, THERE EXISTS A PLANAR STRAIGHT-LINE DRAWING OF G IN WHICH THE VERTICES IN S HAVE THE ASSIGNED x- AND y-COORDINATES.

KNOWN BOUNDS

TREES: $\geq 14n/17$ $\leq 5n/6$ [EVANS et al. 2014]

OUTERPLANAR GRAPHS: $\geq n/2$ [BARBA et al. 2015]

PLANAR GRAPHS OF TREEWIDTH AT MOST THREE: $\geq (n-3)/8$

TRICONNECTED CUBIC PLANAR GRAPHS: $\geq n/4$

PLANAR GRAPHS OF TREEWIDTH K: $\Omega(k^2)$

OPEN PROBLEMS

- CLOSE THE GAP BETWEEN THE $\Omega(n^{0.5})$ AND $O(n^{0.986})$ BOUNDS FOR THE SIZE OF (FREE) COLLINEAR SETS IN GENERAL PLANAR GRAPHS.
- IS IT TRUE THAT EVERY COLLINEAR SET IS FREE? IF NOT, HOW FAR APART ARE THE SIZES OF THESE TWO SETS IN GENERAL PLANAR GRAPHS?
- IS IT TRUE THAT THE MAXIMUM NUMBER OF COLLINEAR INTERNAL VERTICES IN A PLANAR STRAIGHT-LINE DRAWING OF A PLANE 3-TREE WITH m INTERNAL VERTICES IS $\left\lceil \frac{m+2}{3} \right\rceil$ (IN THE WORST CASE)?