Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG
Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems
Institute for Informatics

Drawing Graphs on Few Lines and Few Planes

Steven Chaplick Krzysztof Fleszar
Fabian Lipp Alexander Wolff
Julius-Maximilians-Universität Würzburg, Germany

Alexander Ravsky
National Academy of Science of Ukraine, Lviv, Ukraine

Oleg Verbitsky
Humboldt-Universität zu Berlin, Germany

Our Task

Given a graph G, find a set of planes in 3-space such that there is a crossing-free straight-line drawing of G with all vertices and edges drawn on these planes.
Call the minimum number of planes needed $\rho_3^2(G)$.

Our Task

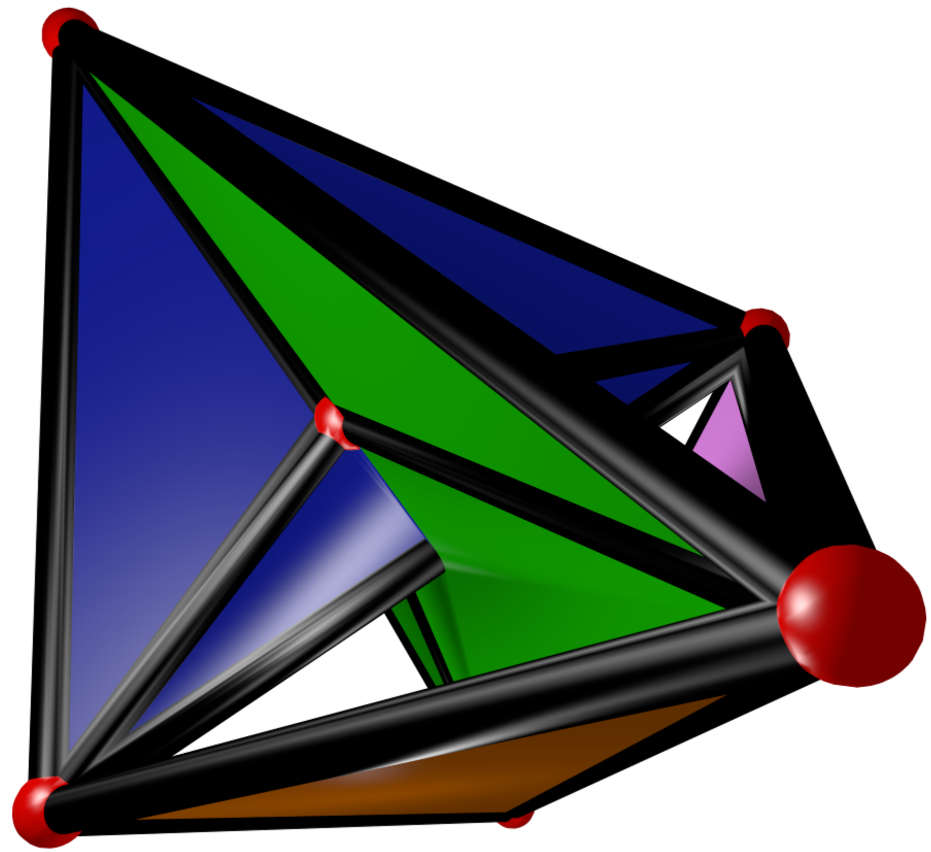
Given a graph G, find a set of planes in 3-space such that there is a crossing-free straight-line drawing of G with all vertices and edges drawn on these planes.
Call the minimum number of planes needed $\rho_3^2(G)$.

$$\rho_3^2(K_6) = ?$$

Our Task

Given a graph G, find a set of planes in 3-space such that there is a crossing-free straight-line drawing of G with all vertices and edges drawn on these planes.
Call the minimum number of planes needed $\rho_3^2(G)$.

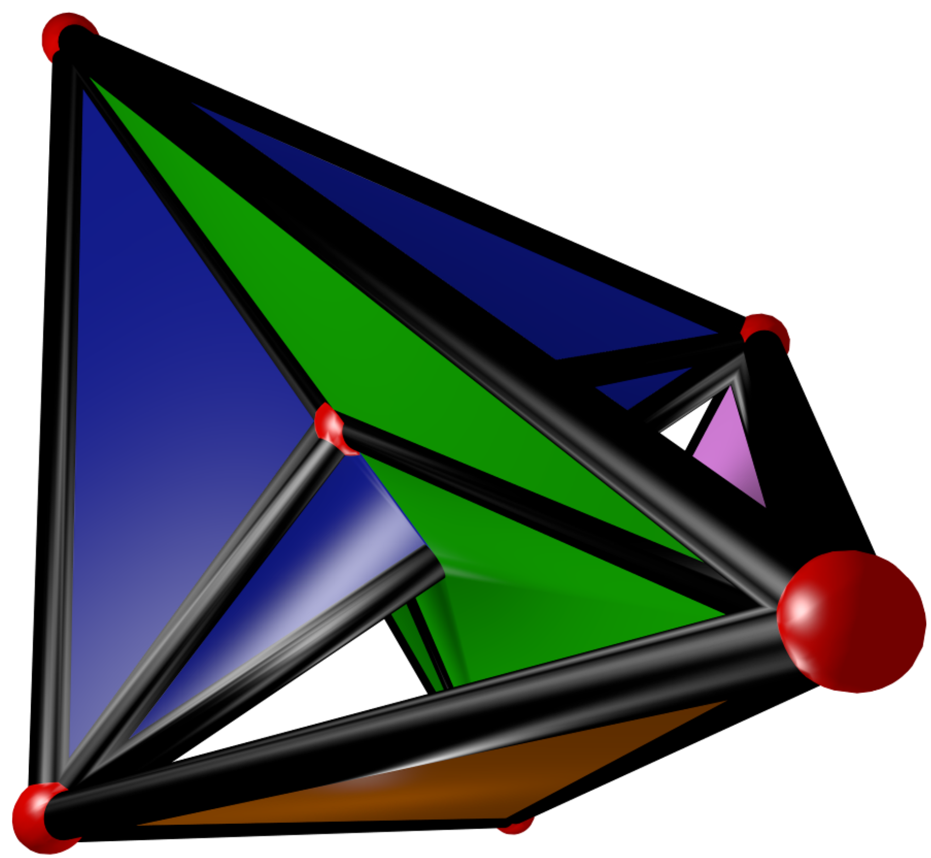
$\rho_3^2(K_6) = 4$



Our Task

Given a graph G, find a set of planes in 3-space such that there is a crossing-free straight-line drawing of G with all vertices and edges drawn on these planes.
Call the minimum number of planes needed $\rho_3^2(G)$.

$\rho_3^2(K_6) = 4$



For planar graphs G, $\rho_3^2(G) = 1$

ρ_3^2 as a parameter for classifying non-planar graphs.

Definitions

Let G be a graph and $1 \leq m < d$.

Affine cover number $\rho_d^m(G)$: minimum number of m-dimensional hyperplanes in $\mathbb{R}^d$ such that G has a crossing-free straight-line drawing that is contained in these planes

Definitions

Let G be a graph and $1 \leq m < d$.

Affine cover number $\rho_d^m(G)$: minimum number of m-dimensional hyperplanes in $\mathbb{R}^d$ such that G has a crossing-free straight-line drawing that is contained in these planes

Weak affine cover number $\pi_d^m(G)$: require only *vertices* to be contained in the planes

Definitions

Let G be a graph and $1 \leq m < d$.

Affine cover number $\rho_d^m(G)$: minimum number of m-dimensional hyperplanes in $\mathbb{R}^d$ such that G has a crossing-free straight-line drawing that is contained in these planes

Weak affine cover number $\pi_d^m(G)$: require only *vertices* to be contained in the planes

Observations

$\rho_d^m = \pi_d^m = 1$ for $m \geq 3$

Definitions

Let G be a graph and $1 \leq m < d$.

Affine cover number $\rho_d^m(G)$: minimum number of m-dimensional hyperplanes in $\mathbb{R}^d$ such that G has a crossing-free straight-line drawing that is contained in these planes

Weak affine cover number $\pi_d^m(G)$: require only *vertices* to be contained in the planes

Observations

$\rho_d^m = \pi_d^m = 1$ for $m \geq 3$ $\rho_d^m = \rho_3^m$ and $\pi_d^m = \pi_3^m$ for $d \geq 3$

Definitions

Let G be a graph and $1 \leq m < d$.

Affine cover number $\rho_d^m(G)$: minimum number of m-dimensional hyperplanes in $\mathbb{R}^d$ such that G has a crossing-free straight-line drawing that is contained in these planes

Weak affine cover number $\pi_d^m(G)$: require only *vertices* to be contained in the planes

Observations

$\rho_d^m = \pi_d^m = 1$ for $m \geq 3$ $\qquad$ $\rho_d^m = \rho_3^m$ and $\pi_d^m = \pi_3^m$ for $d \geq 3$

$\pi_d^m \leq \rho_d^m$

Definitions

Let G be a graph and $1 \leq m < d$.

Affine cover number $\rho_d^m(G)$: minimum number of m-dimensional hyperplanes in $\mathbb{R}^d$ such that G has a crossing-free straight-line drawing that is contained in these planes

Weak affine cover number $\pi_d^m(G)$: require only *vertices* to be contained in the planes

Observations

$\rho_d^m = \pi_d^m = 1$ for $m \geq 3$ $\qquad$ $\rho_d^m = \rho_3^m$ and $\pi_d^m = \pi_3^m$ for $d \geq 3$

$\pi_d^m \leq \rho_d^m$ $\qquad$ $\rho_3^2 \leq \rho_3^1 \leq \rho_2^1$ $\qquad$ $\pi_3^2 \leq \pi_3^1 \leq \pi_2^1$

Definitions

Let G be a graph and $1 \leq m < d$.

Affine cover number $\rho_d^m(G)$: minimum number of m-dimensional hyperplanes in $\mathbb{R}^d$ such that G has a crossing-free straight-line drawing that is contained in these planes

Weak affine cover number $\pi_d^m(G)$: require only *vertices* to be contained in the planes

Observations

$\rho_d^m = \pi_d^m = 1$ for $m \geq 3$ $\qquad$ $\rho_d^m = \rho_3^m$ and $\pi_d^m = \pi_3^m$ for $d \geq 3$

$\pi_d^m \leq \rho_d^m$ $\qquad$ $\rho_3^2 \leq \rho_3^1 \leq \rho_2^1$ $\qquad$ $\pi_3^2 \leq \pi_3^1 \leq \pi_2^1$

Interesting cases

- Line cover numbers in 2D and 3D: ρ_2^1, ρ_3^1, π_2^1, π_3^1

Definitions

Let G be a graph and $1 \leq m < d$.

Affine cover number $\rho_d^m(G)$: minimum number of m-dimensional hyperplanes in $\mathbb{R}^d$ such that G has a crossing-free straight-line drawing that is contained in these planes

Weak affine cover number $\pi_d^m(G)$: require only *vertices* to be contained in the planes

Observations

$\rho_d^m = \pi_d^m = 1$ for $m \geq 3$ $\qquad$ $\rho_d^m = \rho_3^m$ and $\pi_d^m = \pi_3^m$ for $d \geq 3$

$\pi_d^m \leq \rho_d^m$ $\qquad$ $\rho_3^2 \leq \rho_3^1 \leq \rho_2^1$ $\qquad$ $\pi_3^2 \leq \pi_3^1 \leq \pi_2^1$

Interesting cases

- Line cover numbers in 2D and 3D: ρ_2^1, ρ_3^1, π_2^1, π_3^1
- Plane cover numbers: ρ_3^2, π_3^2

Our Results

Complexity: Computing ρ^1_2, ρ^1_3, ρ^2_3, π^1_3, and π^2_3 is NP-hard.

Our Results

Complexity: Computing ρ_2^1, ρ_3^1, ρ_3^2, π_3^1, and π_3^2 is NP-hard.

Our Results

Complexity: Computing ρ_2^1, ρ_3^1, ρ_3^2, π_3^1, and π_3^2 is NP-hard.

π_2^1	ρ_2^1
π_3^1	ρ_3^1
π_3^2	ρ_3^2

Our Results

Complexity: Computing ρ_2^1, ρ_3^1, ρ_3^2, π_3^1, and π_3^2 is NP-hard.

Upper bound for outerplanar graphs	π_2^1	ρ_2^1	Relations to other graph parameters
Relations to other graph parameters	π_3^1	ρ_3^1	Lower bound
	π_3^2	ρ_3^2	Bounds for K_n

Our Results

Complexity: Computing ρ_2^1, ρ_3^1, ρ_3^2, π_3^1, and π_3^2 is NP-hard.

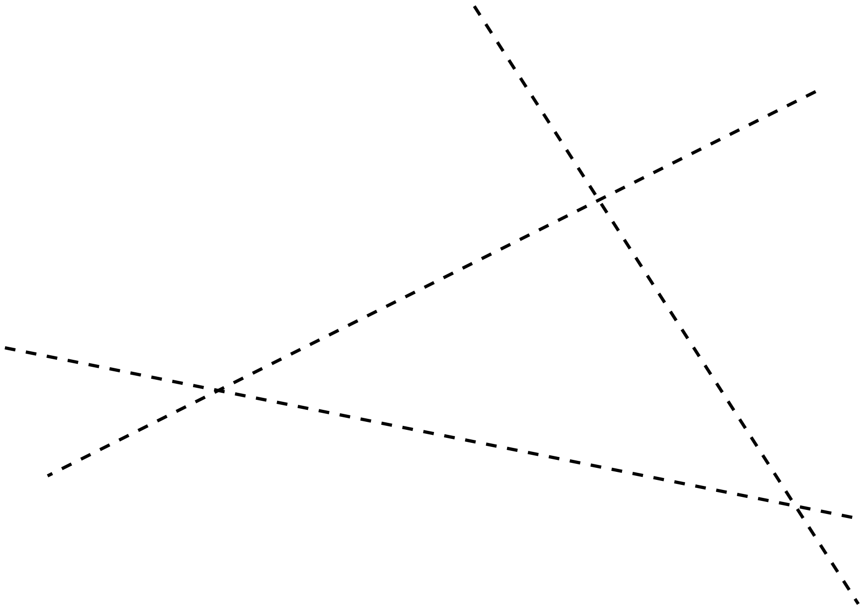
Upper bound for outerplanar graphs	π_2^1	ρ_2^1	Relations to other graph parameters
Relations to other graph parameters	π_3^1, π_3^2	ρ_3^1	Lower bound
		ρ_3^2	Bounds for K_n

Relations to Other Parameters

- $\chi(G)/2 \leq \pi_3^1(G) \leq \chi(G)$

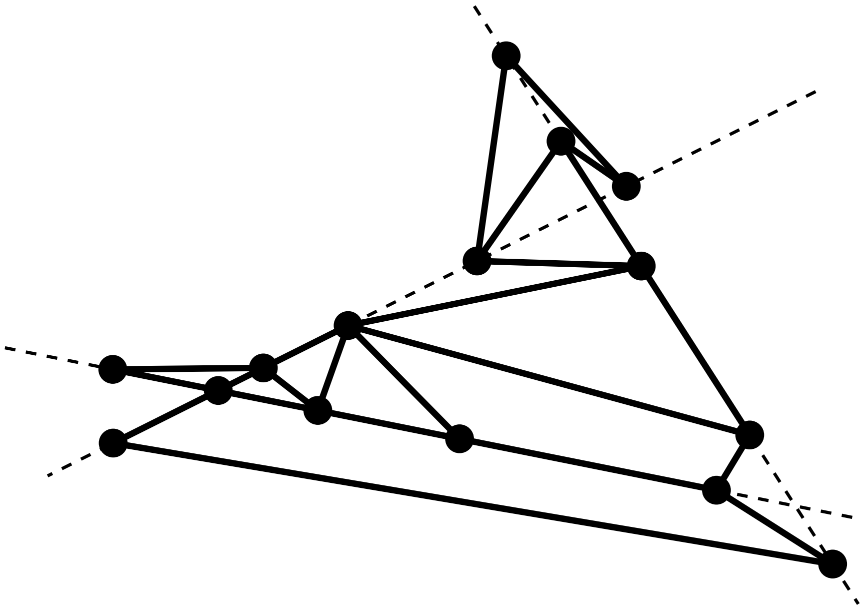
Relations to Other Parameters

- $\chi(G)/2 \le \pi_3^1(G) \le \chi(G)$



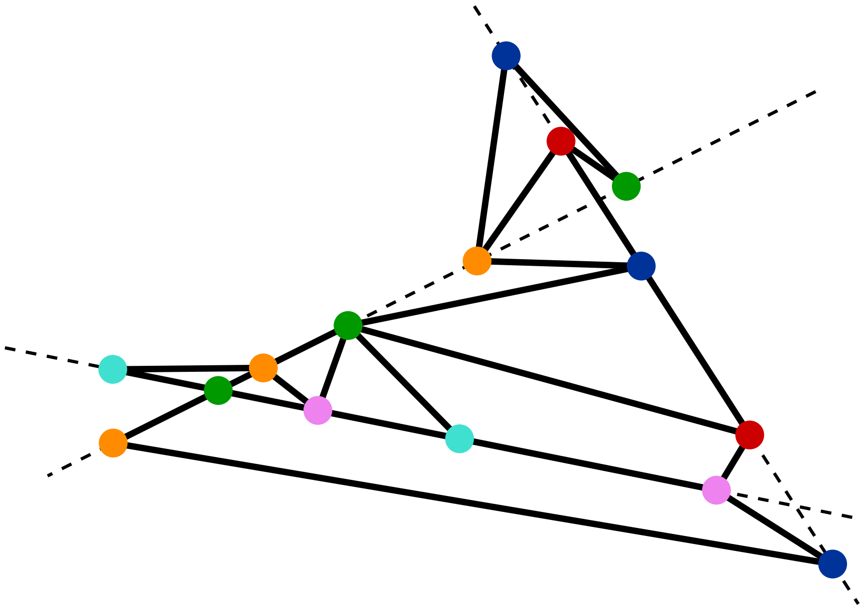
Relations to Other Parameters

- $\chi(G)/2 \leq \pi^1_3(G) \leq \chi(G)$



Relations to Other Parameters

- $\chi(G)/2 \leq \pi_3^1(G) \leq \chi(G)$



Relations to Other Parameters

- $\chi(G)/2 \leq \pi_3^1(G) \leq \chi(G)$
- $\chi(G)/4 \leq \pi_3^2(G) \leq \chi(G)$

Relations to Other Parameters

- $\chi(G)/2 \leq \pi_3^1(G) \leq \chi(G)$
- $\chi(G)/4 \leq \pi_3^2(G) \leq \chi(G)$
- $\pi_3^1(G) = \mathrm{lva}(G)$
  linear vertex arboricity: smallest partition of V such that each set induces a linear forest

Relations to Other Parameters

- $\chi(G)/2 \leq \pi_3^1(G) \leq \chi(G)$
- $\chi(G)/4 \leq \pi_3^2(G) \leq \chi(G)$
- $\pi_3^1(G) = \mathrm{lva}(G)$
  linear vertex arboricity: smallest partition of V such that each set induces a linear forest

$$\pi_3^1(G) \leq \begin{cases} 3 \text{ for planar } G & \text{[Poh, 1990]} \\ 2 \text{ for outerplanar } G & \text{[Broere, Mynhardt, 1985]} \\ \Delta/2 + 1 \text{ for connected } G & \text{[Matsumoto, 1990]} \end{cases}$$

Relations to Other Parameters

- $\chi(G)/2 \leq \pi_3^1(G) \leq \chi(G)$
- $\chi(G)/4 \leq \pi_3^2(G) \leq \chi(G)$
- $\pi_3^1(G) = \mathrm{lva}(G)$
  linear vertex arboricity: smallest partition of V such that each set induces a linear forest

  $$\pi_3^1(G) \leq \begin{cases} 3 \text{ for planar } G & \text{[Poh, 1990]} \\ 2 \text{ for outerplanar } G & \text{[Broere, Mynhardt, 1985]} \\ \Delta/2 + 1 \text{ for connected } G & \text{[Matsumoto, 1990]} \end{cases}$$

- $\pi_3^2(G) = \mathrm{vt}(G)$
  vertex thickness: smallest partition of V such that each set induces a planar graph

Relations to Other Parameters

For planar, connected graphs G:

$$\operatorname{slope}(G) \leq \rho^1_2(G) \leq \operatorname{segm}(G)$$

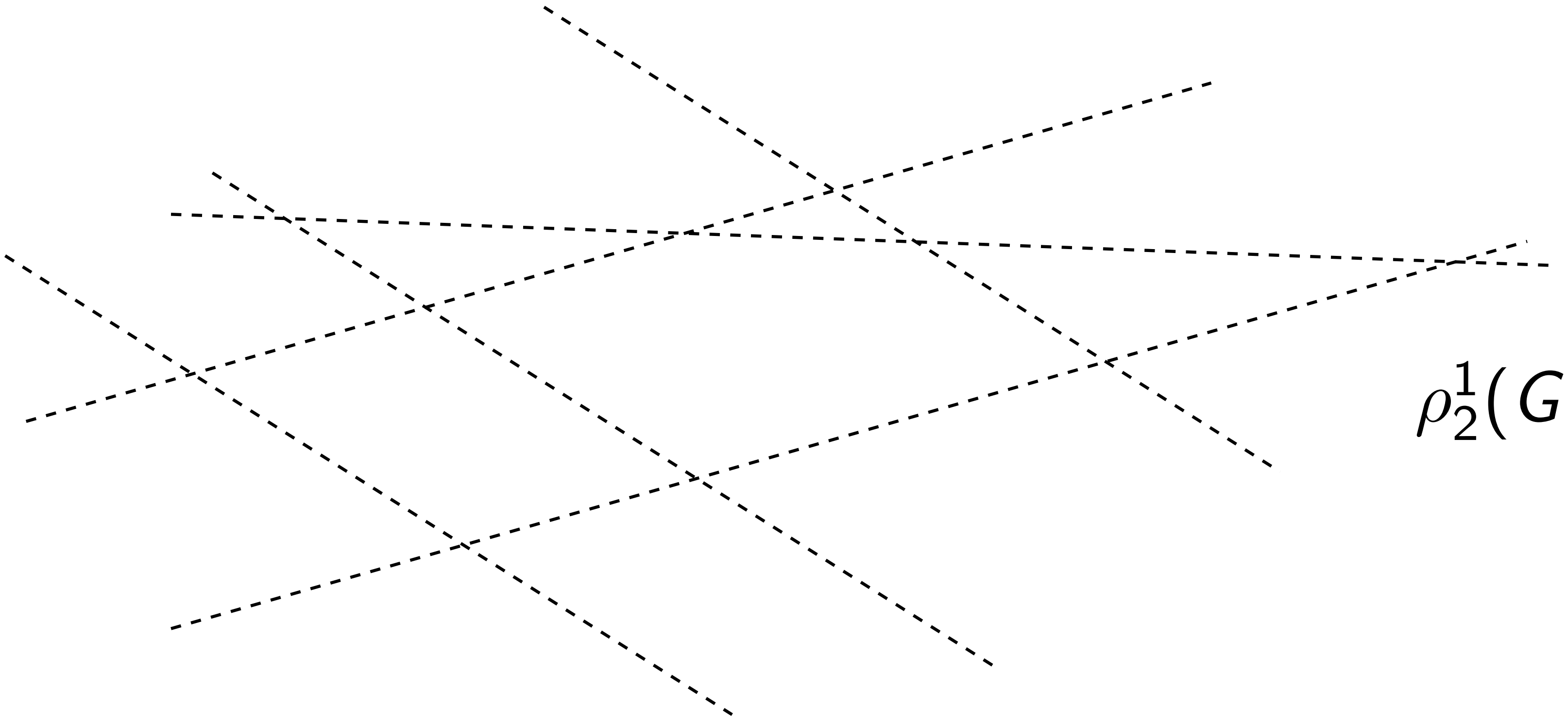
Relations to Other Parameters

For planar, connected graphs G:

minimum number of different slopes needed to draw G

[Dujmović et al. 2007]

$\text{slope}(G) \leq \rho_2^1(G) \leq \text{segm}(G)$



$\rho_2^1(G) \leq 6$

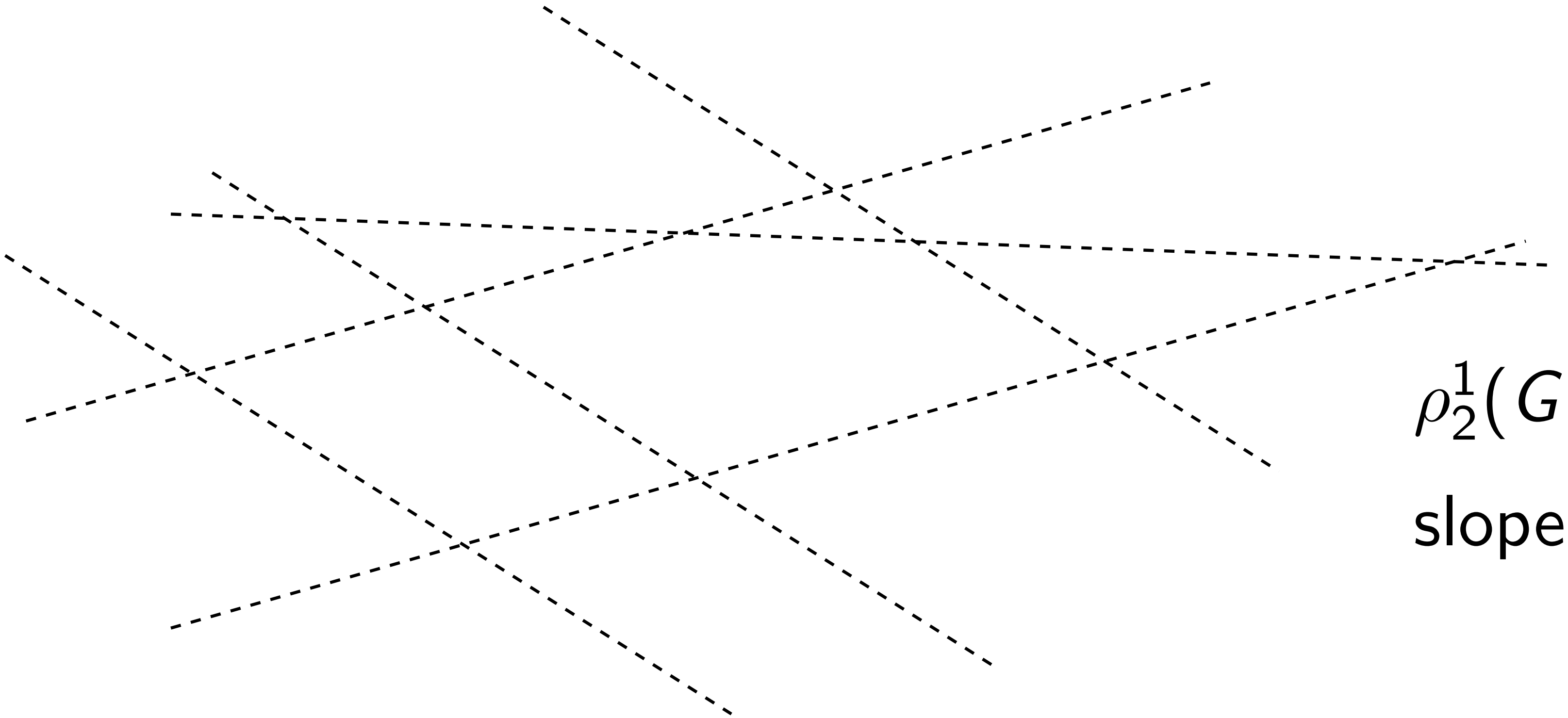
Relations to Other Parameters

For planar, connected graphs G:

minimum number of different slopes needed to draw G

[Dujmović et al. 2007]

$$\mathrm{slope}(G) \leq \rho_2^1(G) \leq \mathrm{segm}(G)$$



$\rho_2^1(G) \leq 6$

$\mathrm{slope}(G) \leq 3$

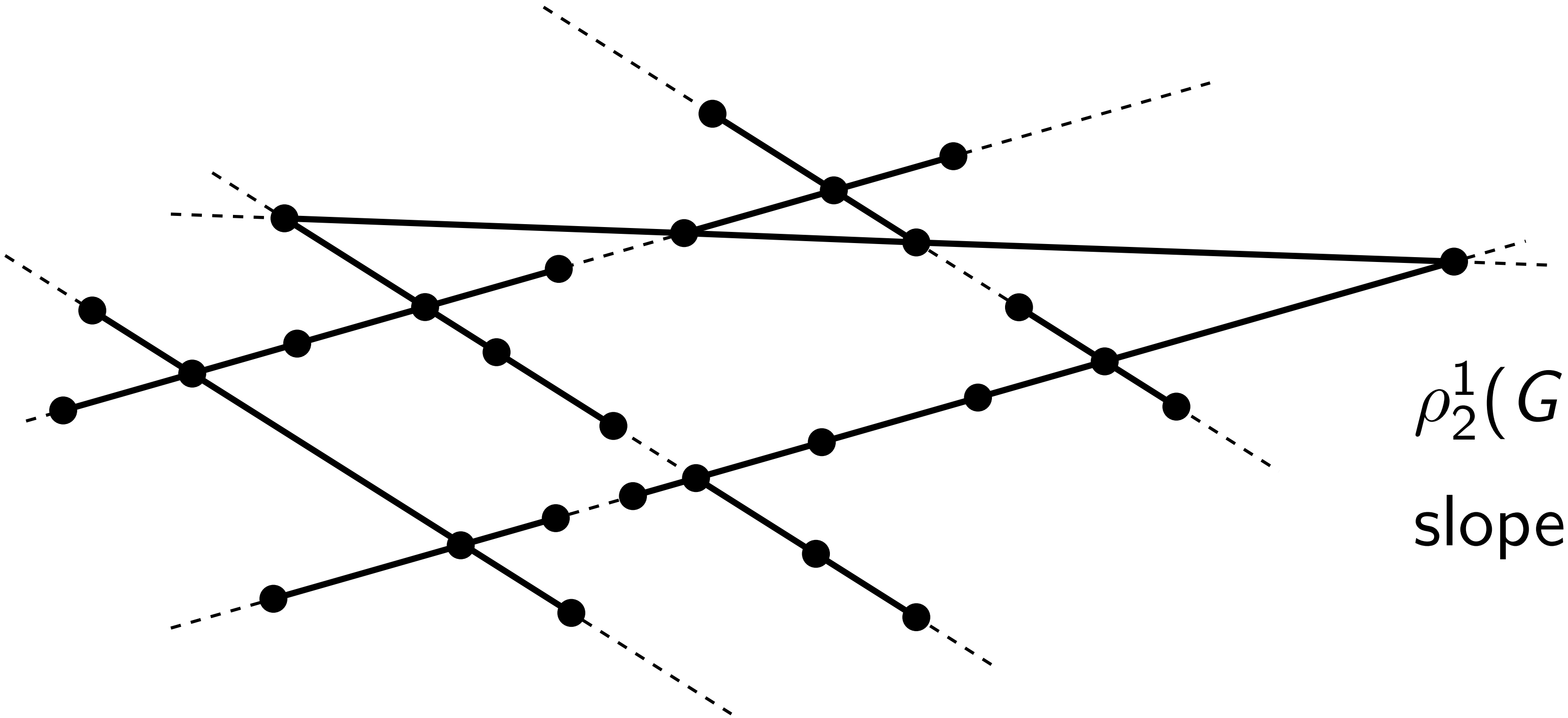
Relations to Other Parameters

For planar, connected graphs G:

minimum number of different slopes needed to draw G

[Dujmović et al. 2007]

$\mathrm{slope}(G) \leq \rho_2^1(G) \leq \mathrm{segm}(G)$



$\rho_2^1(G) \leq 6$

$\mathrm{slope}(G) \leq 3$

Relations to Other Parameters

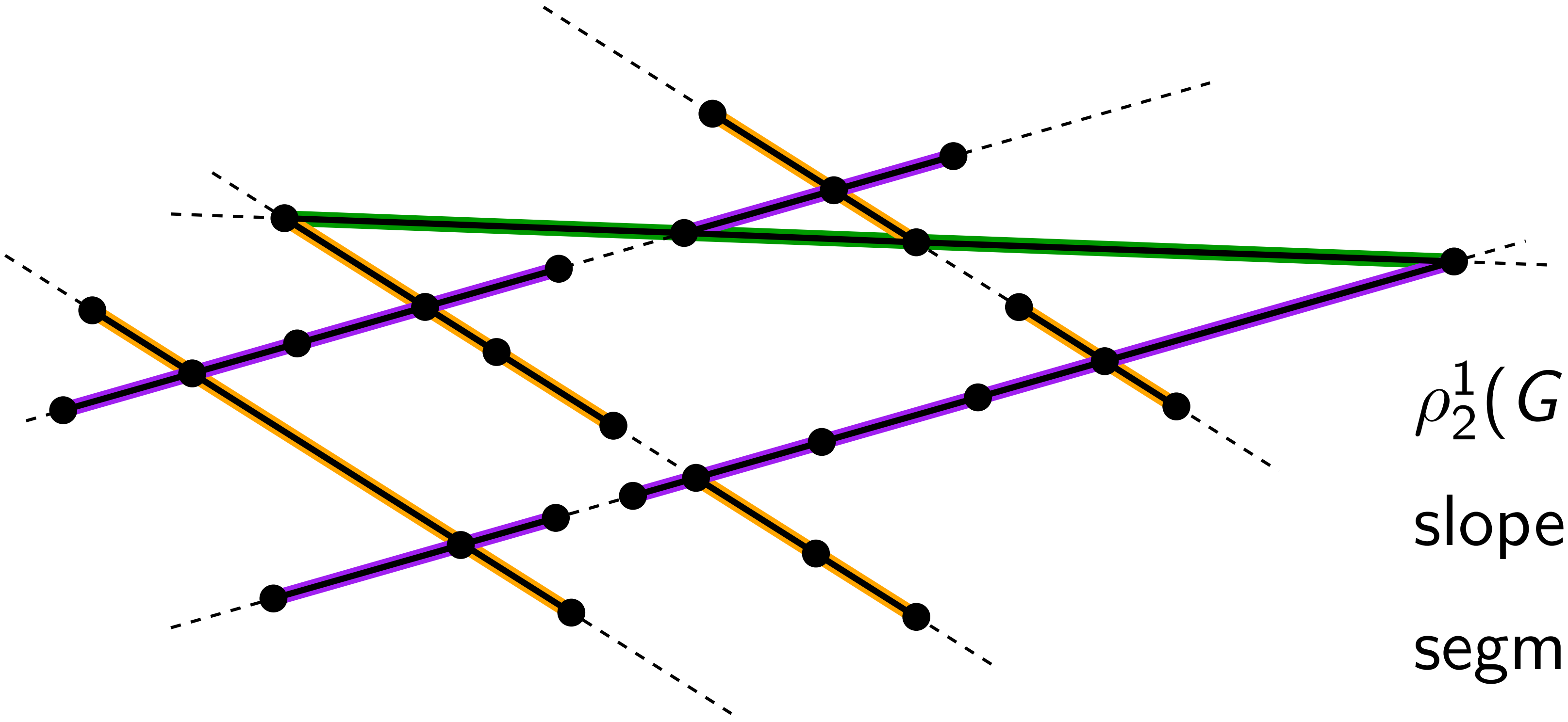
For planar, connected graphs G:

minimum number of different slopes needed to draw G

[Dujmović et al. 2007]

$$\mathrm{slope}(G) \leq \rho^1_2(G) \leq \mathrm{segm}(G)$$

minimum number of line segments needed to draw G



$\rho^1_2(G) \leq 6$

$\mathrm{slope}(G) \leq 3$

$\mathrm{segm}(G) \leq 10$

Relations to Other Parameters

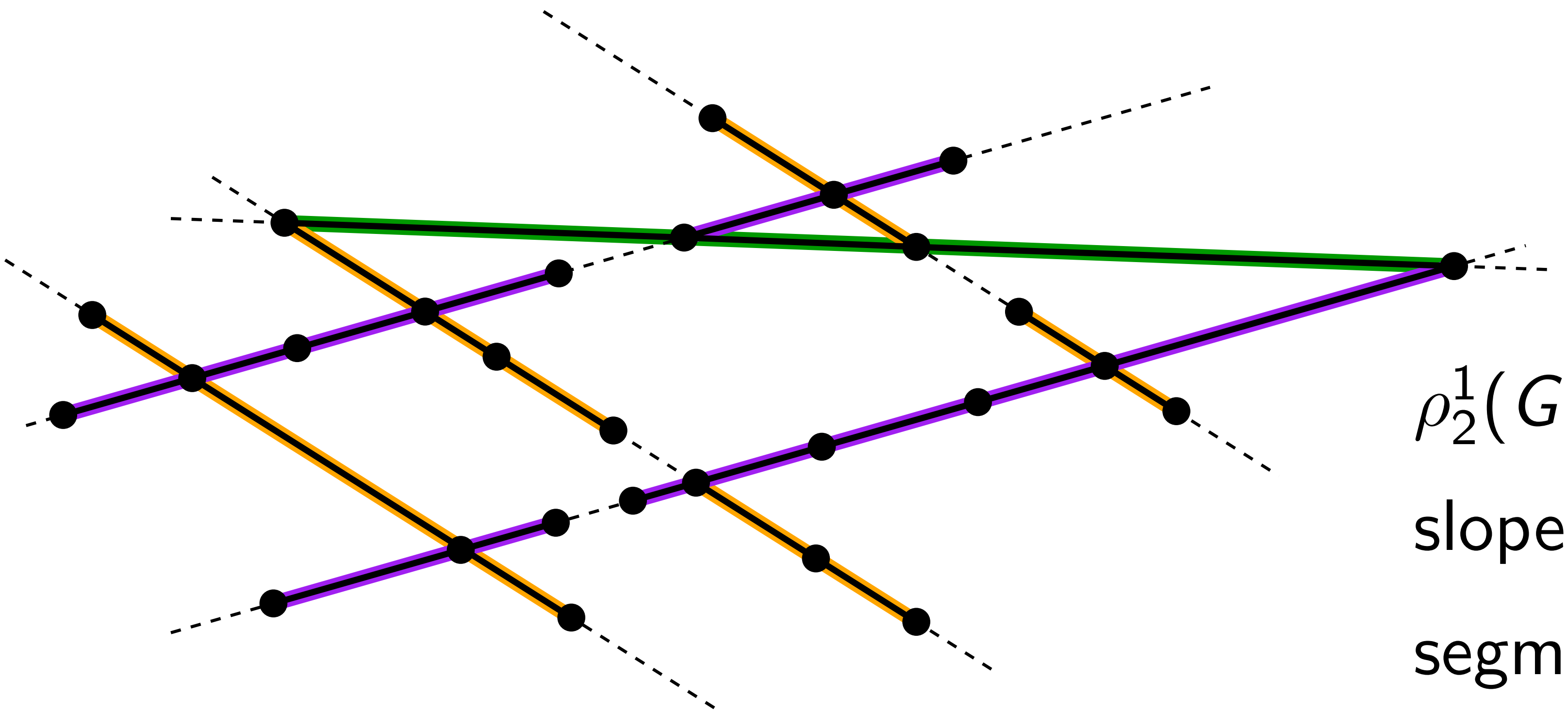
For planar, connected graphs G:

minimum number of different slopes needed to draw G

[Dujmović et al. 2007]

$$\mathrm{slope}(G) \leq \rho^1_2(G) \leq \mathrm{segm}(G) = O(\rho^1_2(G)^2)$$

minimum number of line segments needed to draw G



$\rho^1_2(G) \leq 6$

$\mathrm{slope}(G) \leq 3$

$\mathrm{segm}(G) \leq 10$

Our Results

Complexity: Computing ρ^1_2, ρ^1_3, ρ^2_3, π^1_3, and π^2_3 is NP-hard.

Upper bound for outerplanar graphs — π^1_2

Relations to other graph parameters — π^1_3, π^2_3

ρ^1_2 — Relations to other graph parameters

ρ^1_3 — Lower bound

ρ^2_3 — Bounds for K_n

Complete Graphs K_n

- $\pi_3^1(K_n) =$

Complete Graphs K_n

- $\pi_3^1(K_n) =$ *Idea:* Draw K_n in general position. Place lines to cover two vertices each.

Complete Graphs K_n

- $\pi_3^1(K_n) = \lceil n/2 \rceil$. *Idea:* Draw K_n in general position. Place lines to cover two vertices each.

Complete Graphs K_n

- $\pi_3^1(K_n) = \lceil n/2 \rceil$. *Idea:* Draw K_n in general position. Place lines to cover two vertices each.

- $\pi_3^2(K_n) =$

Complete Graphs K_n

- $\pi_3^1(K_n) = \lceil n/2 \rceil$. *Idea:* Draw K_n in general position. Place lines to cover two vertices each.
- $\pi_3^2(K_n) =$ *Idea:* Place copies of K_4 on parallel planes and wiggle to avoid crossings.

Complete Graphs K_n

- $\pi_3^1(K_n) = \lceil n/2 \rceil$. *Idea:* Draw K_n in general position. Place lines to cover two vertices each.
- $\pi_3^2(K_n) = \lceil n/4 \rceil$. *Idea:* Place copies of K_4 on parallel planes and wiggle to avoid crossings.

Complete Graphs K_n

- $\pi_3^1(K_n) = \lceil n/2 \rceil$. *Idea:* Draw K_n in general position. Place lines to cover two vertices each.

- $\pi_3^2(K_n) = \lceil n/4 \rceil$. *Idea:* Place copies of K_4 on parallel planes and wiggle to avoid crossings.

- $\rho_3^1(K_n) =$

Complete Graphs K_n

- $\pi_3^1(K_n) = \lceil n/2 \rceil$. *Idea:* Draw K_n in general position. Place lines to cover two vertices each.

- $\pi_3^2(K_n) = \lceil n/4 \rceil$. *Idea:* Place copies of K_4 on parallel planes and wiggle to avoid crossings.

- $\rho_3^1(K_n) =$ *Idea:* Any line contains at most one edge.

Complete Graphs K_n

- $\pi_3^1(K_n) = \lceil n/2 \rceil$. *Idea:* Draw K_n in general position. Place lines to cover two vertices each.
- $\pi_3^2(K_n) = \lceil n/4 \rceil$. *Idea:* Place copies of K_4 on parallel planes and wiggle to avoid crossings.
- $\rho_3^1(K_n) = \binom{n}{2}$. *Idea:* Any line contains at most one edge.

Complete Graphs K_n

- $\pi_3^1(K_n) = \lceil n/2 \rceil$. *Idea:* Draw K_n in general position. Place lines to cover two vertices each.

- $\pi_3^2(K_n) = \lceil n/4 \rceil$. *Idea:* Place copies of K_4 on parallel planes and wiggle to avoid crossings.

- $\rho_3^1(K_n) = \binom{n}{2}$. *Idea:* Any line contains at most one edge.

- $\frac{n(n-1)}{4 \cdot 3} \leq c(K_n, K_4) \leq \rho_3^2(K_n) \leq c(K_n, K_3) = \frac{n^2}{6} + O(n)$

Complete Graphs K_n

- $\pi_3^1(K_n) = \lceil n/2 \rceil$. *Idea:* Draw K_n in general position. Place lines to cover two vertices each.
- $\pi_3^2(K_n) = \lceil n/4 \rceil$. *Idea:* Place copies of K_4 on parallel planes and wiggle to avoid crossings.
- $\rho_3^1(K_n) = \binom{n}{2}$. *Idea:* Any line contains at most one edge.
- $\frac{n(n-1)}{4\cdot 3} \leq c(K_n, K_4) \leq \rho_3^2(K_n) \leq c(K_n, K_3) = \frac{n^2}{6} + O(n)$

minimum number of copies of K_4 to cover all edges of K_n

Complete Graphs K_n

- $\pi_3^1(K_n) = \lceil n/2 \rceil$. *Idea:* Draw K_n in general position. Place lines to cover two vertices each.
- $\pi_3^2(K_n) = \lceil n/4 \rceil$. *Idea:* Place copies of K_4 on parallel planes and wiggle to avoid crossings.
- $\rho_3^1(K_n) = \binom{n}{2}$. *Idea:* Any line contains at most one edge.
- $\frac{n(n-1)}{4 \cdot 3} \leq c(K_n, K_4) \leq \rho_3^2(K_n) \leq c(K_n, K_3) = \frac{n^2}{6} + O(n)$

at most one K_4 on any plane

Complete Graphs K_n

- $\pi_3^1(K_n) = \lceil n/2 \rceil$. *Idea:* Draw K_n in general position. Place lines to cover two vertices each.
- $\pi_3^2(K_n) = \lceil n/4 \rceil$. *Idea:* Place copies of K_4 on parallel planes and wiggle to avoid crossings.
- $\rho_3^1(K_n) = \binom{n}{2}$. *Idea:* Any line contains at most one edge.
- $\frac{n(n-1)}{4 \cdot 3} \leq c(K_n, K_4) \leq \rho_3^2(K_n) \leq c(K_n, K_3) = \frac{n^2}{6} + O(n)$

at most one K_4 on any plane

distribute $n(n-1)/2$ edges on planes containing at most $4 \cdot 3/2$ edges each

Complete Graphs K_n

- $\pi_3^1(K_n) = \lceil n/2 \rceil$. *Idea:* Draw K_n in general position. Place lines to cover two vertices each.

- $\pi_3^2(K_n) = \lceil n/4 \rceil$. *Idea:* Place copies of K_4 on parallel planes and wiggle to avoid crossings.

- $\rho_3^1(K_n) = \binom{n}{2}$. *Idea:* Any line contains at most one edge.

- $\frac{n(n-1)}{4\cdot 3} \le c(K_n, K_4) \le \rho_3^2(K_n) \le c(K_n, K_3) = \frac{n^2}{6} + O(n)$

use one plane for each K_3

Complete Graphs K_n

- $\pi_3^1(K_n) = \lceil n/2 \rceil$. *Idea:* Draw K_n in general position. Place lines to cover two vertices each.

- $\pi_3^2(K_n) = \lceil n/4 \rceil$. *Idea:* Place copies of K_4 on parallel planes and wiggle to avoid crossings.

- $\rho_3^1(K_n) = \binom{n}{2}$. *Idea:* Any line contains at most one edge.

- $\frac{n(n-1)}{4 \cdot 3} \leq c(K_n, K_4) \leq \rho_3^2(K_n) \leq c(K_n, K_3) = \frac{n^2}{6} + O(n)$

use one plane for each K_3

Results on Steiner triple systems imply that there is a set of copies of K_3 that cover each edge exactly once iff $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$

[Kirkman, 1847]

Our Results

Complexity: Computing ρ^1_2, ρ^1_3, ρ^2_3, π^1_3, and π^2_3 is NP-hard.

Upper bound for outerplanar graphs — π^1_2

ρ^1_2 — Relations to other graph parameters

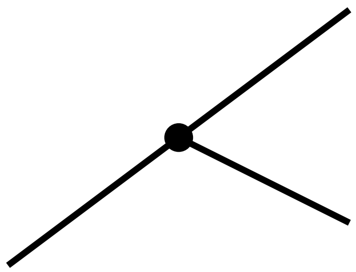
Relations to other graph parameters — π^1_3, π^2_3

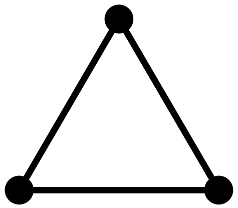
ρ^1_3 — Lower bound

ρ^2_3 — Bounds for K_n

Lower Bound for $\rho_3^1(G)$

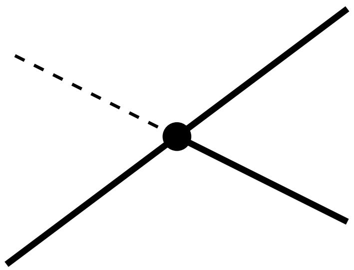
Essential vertex: degree ≥ 3 or belongs to a K_3

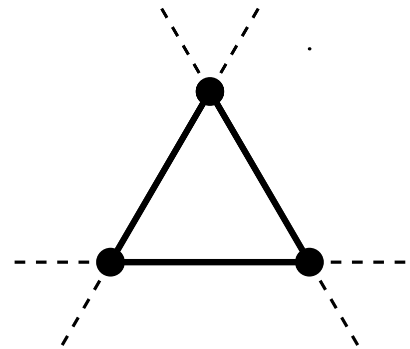




Lower Bound for $\rho_3^1(G)$

Essential vertex: degree ≥ 3 or belongs to a K_3

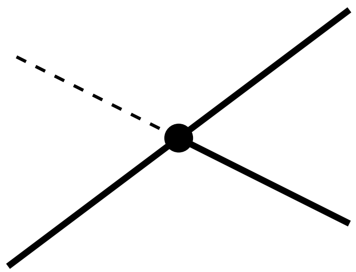


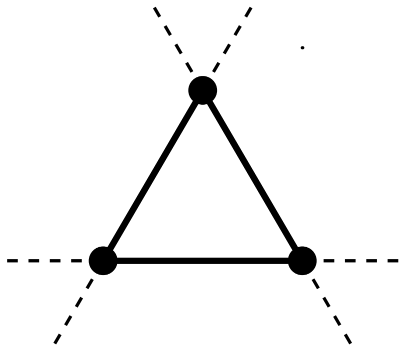


- Essential vertices on intersections of at least two lines

Lower Bound for $\rho_3^1(G)$

Essential vertex: degree ≥ 3 or belongs to a K_3

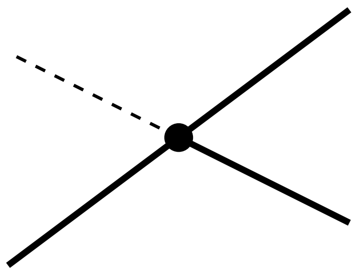


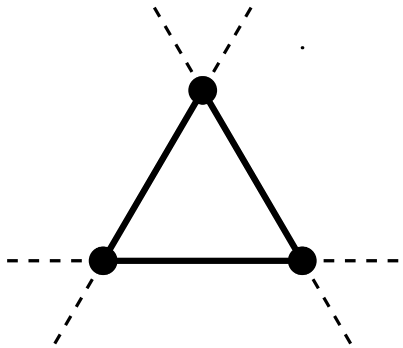


- Essential vertices on intersections of at least two lines
- $\mathrm{es}(G)$: number of essential vertices in G
- $\mathrm{es}(G) \leq \binom{\rho_3^1(G)}{2}$

Lower Bound for $\rho_3^1(G)$

Essential vertex: degree ≥ 3 or belongs to a K_3

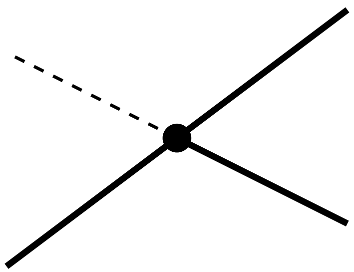


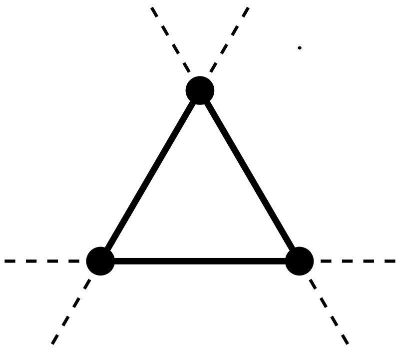


- Essential vertices on intersections of at least two lines
- $\mathrm{es}(G)$: number of essential vertices in G
- $\mathrm{es}(G) \leq \binom{\rho_3^1(G)}{2}$ $\Rightarrow$ $\rho_3^1(G) \geq \sqrt{2\,\mathrm{es}(G)}$

Lower Bound for $\rho_3^1(G)$

Essential vertex: degree ≥ 3 or belongs to a K_3





- Essential vertices on intersections of at least two lines
- $\mathrm{es}(G)$: number of essential vertices in G
- $\mathrm{es}(G) \leq \binom{\rho_3^1(G)}{2}$ $\Rightarrow$ $\rho_3^1(G) \geq \sqrt{2\,\mathrm{es}(G)}$

Additionally: $\rho_3^1(G) \geq \mathrm{tw}(G)/3$

Our Results

Complexity: Computing ρ_2^1, ρ_3^1, ρ_3^2, π_3^1, and π_3^2 is NP-hard.

Upper bound for outerplanar graphs π_2^1

ρ_2^1 Relations to other graph parameters

Relations to other graph parameters: π_3^1, π_3^2

ρ_3^1 Lower bound

ρ_3^2 Bounds for K_n

π_2^1 for Outerplanar Graphs

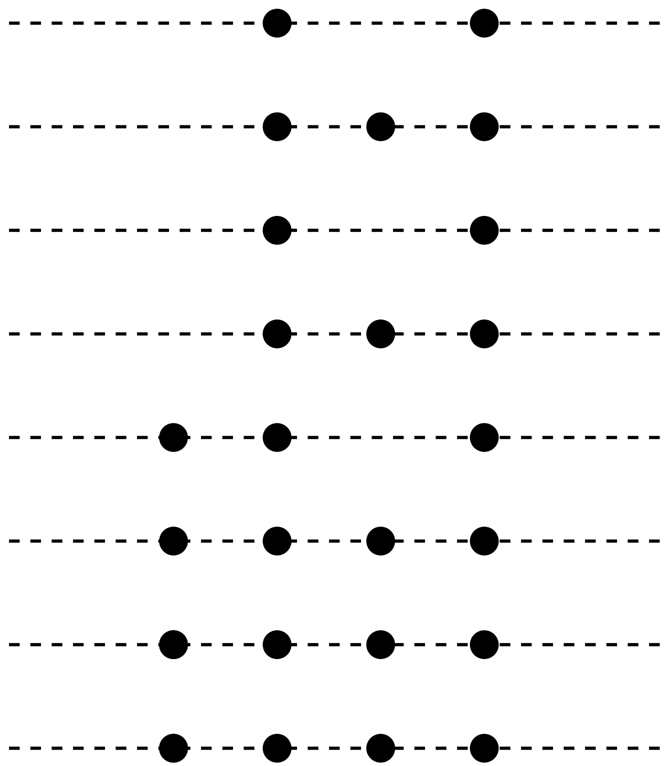
Outerplanar graphs are track drawable.

[Felsner, Liotta, Wismath, JGAA 2003]

π_2^1 for Outerplanar Graphs

Outerplanar graphs are track drawable.

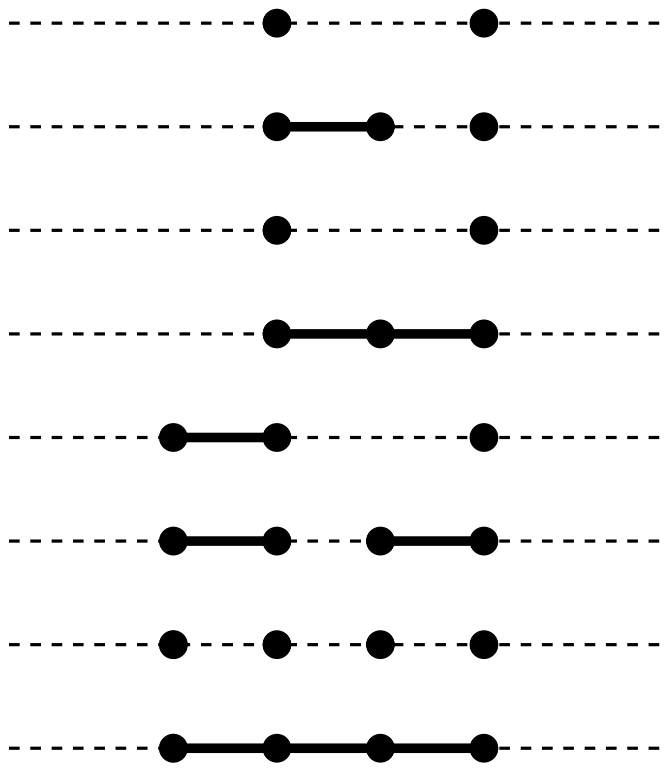
[Felsner, Liotta, Wismath, JGAA 2003]



π_2^1 for Outerplanar Graphs

Outerplanar graphs are track drawable.

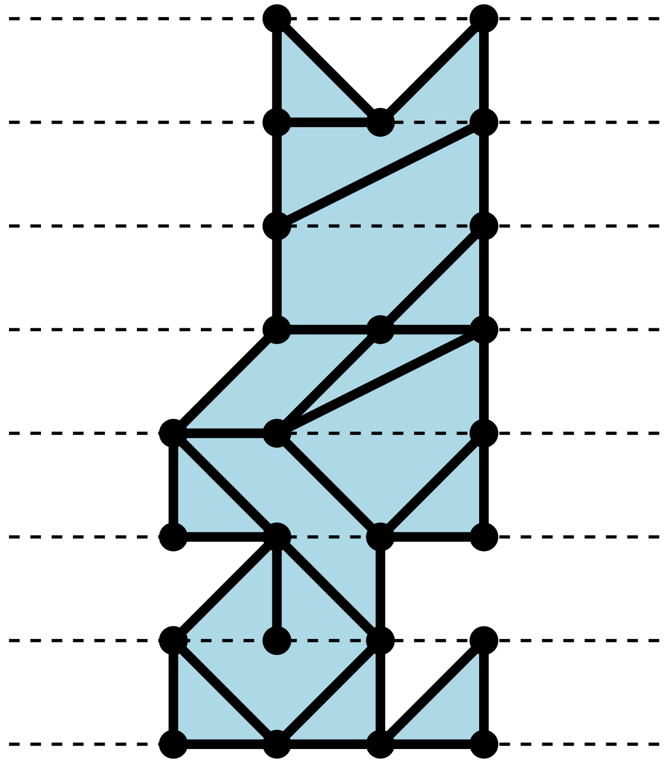
[Felsner, Liotta, Wismath, JGAA 2003]



π_2^1 for Outerplanar Graphs

Outerplanar graphs are track drawable.

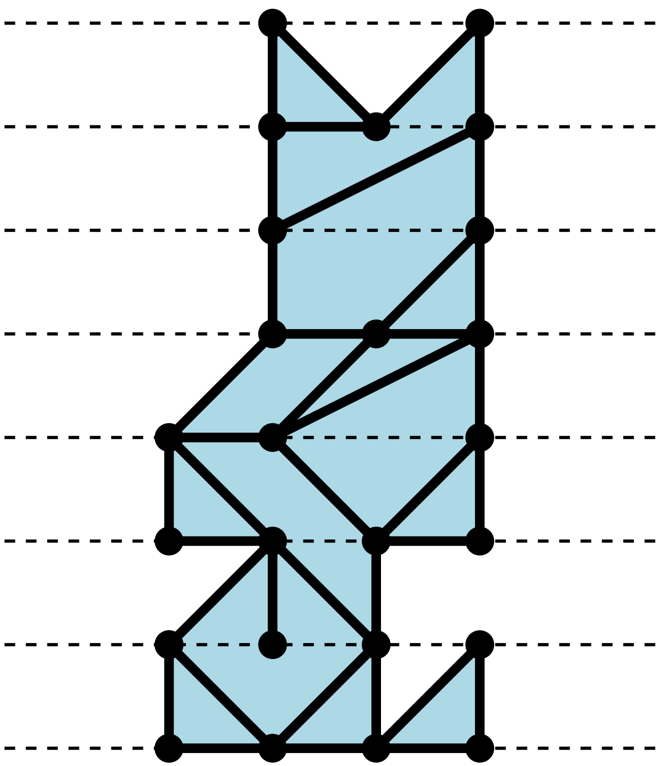
[Felsner, Liotta, Wismath, JGAA 2003]

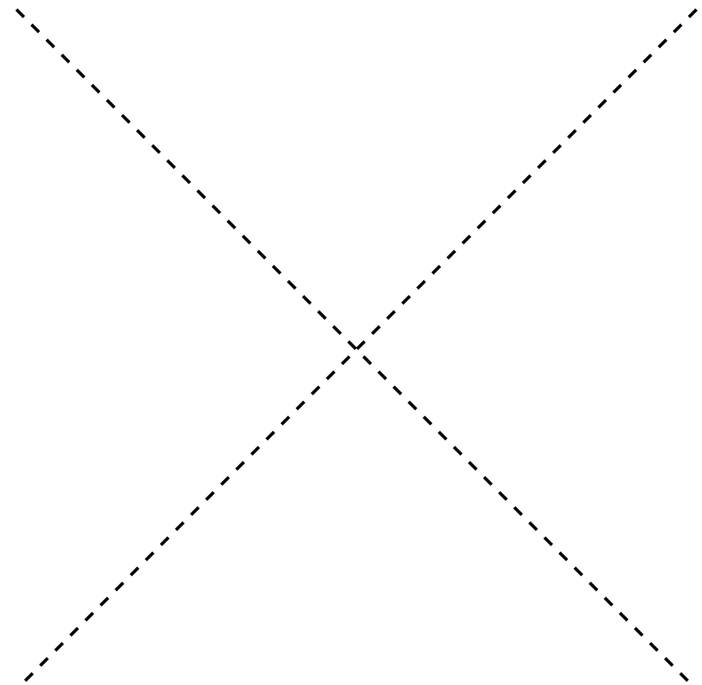


π_2^1 for Outerplanar Graphs

Outerplanar graphs are track drawable.

[Felsner, Liotta, Wismath, JGAA 2003]

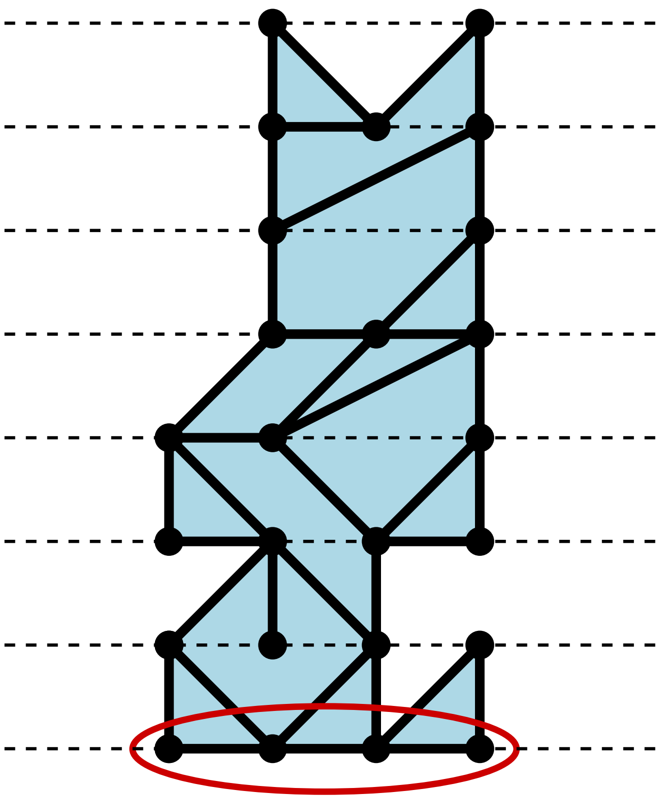


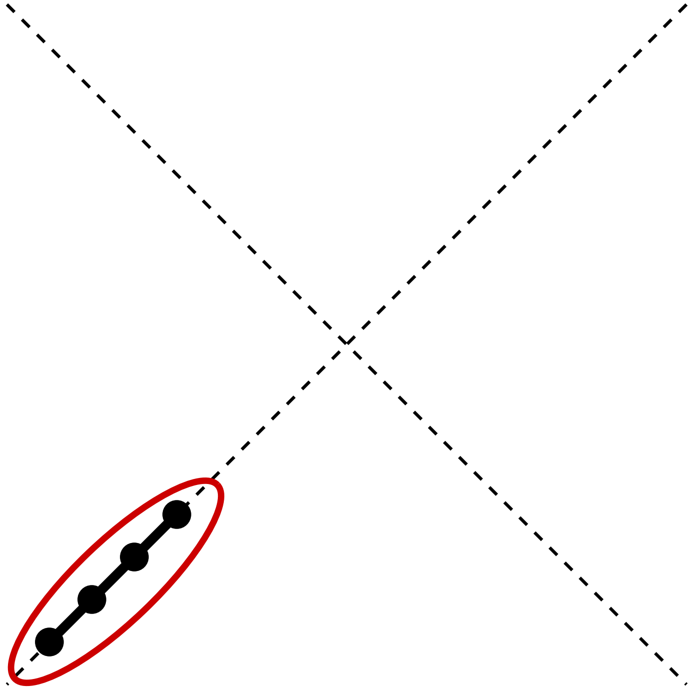


π_2^1 for Outerplanar Graphs

Outerplanar graphs are track drawable.

[Felsner, Liotta, Wismath, JGAA 2003]

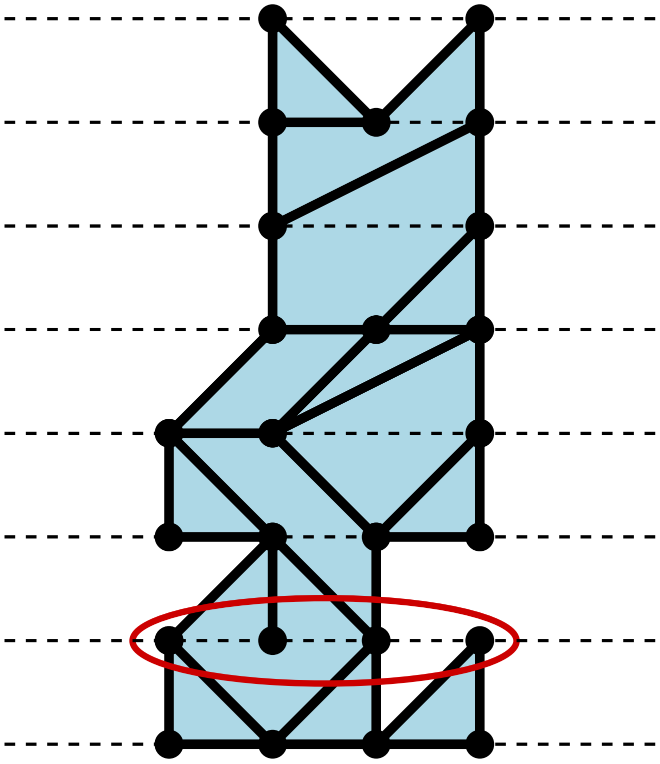


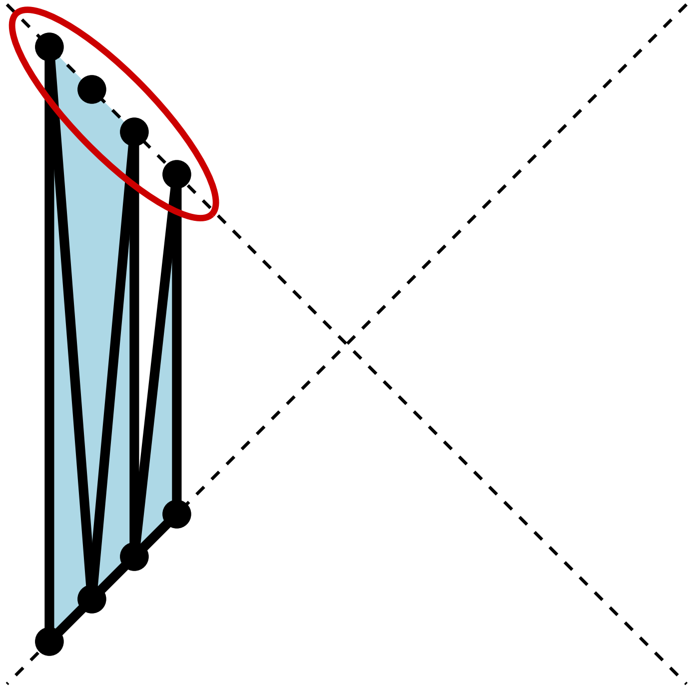


π_2^1 for Outerplanar Graphs

Outerplanar graphs are track drawable.

[Felsner, Liotta, Wismath, JGAA 2003]

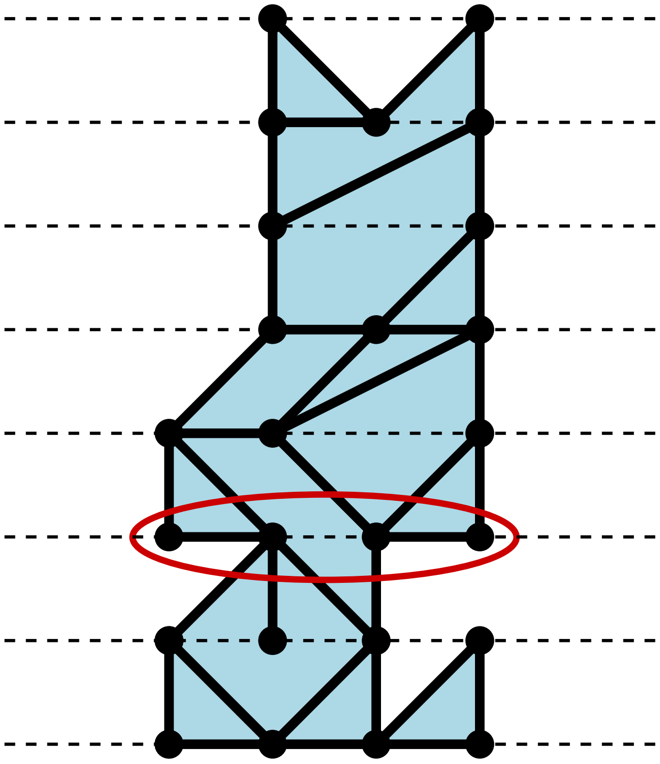


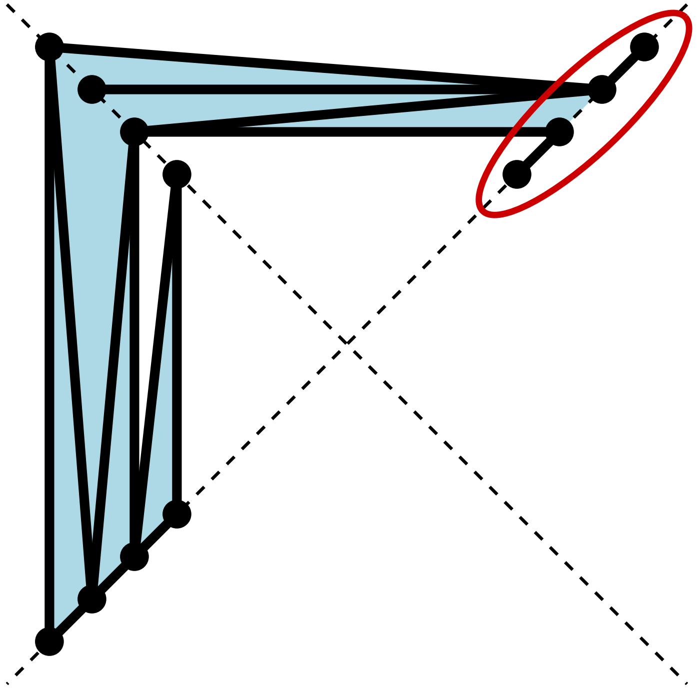


π_2^1 for Outerplanar Graphs

Outerplanar graphs are track drawable.

[Felsner, Liotta, Wismath, JGAA 2003]

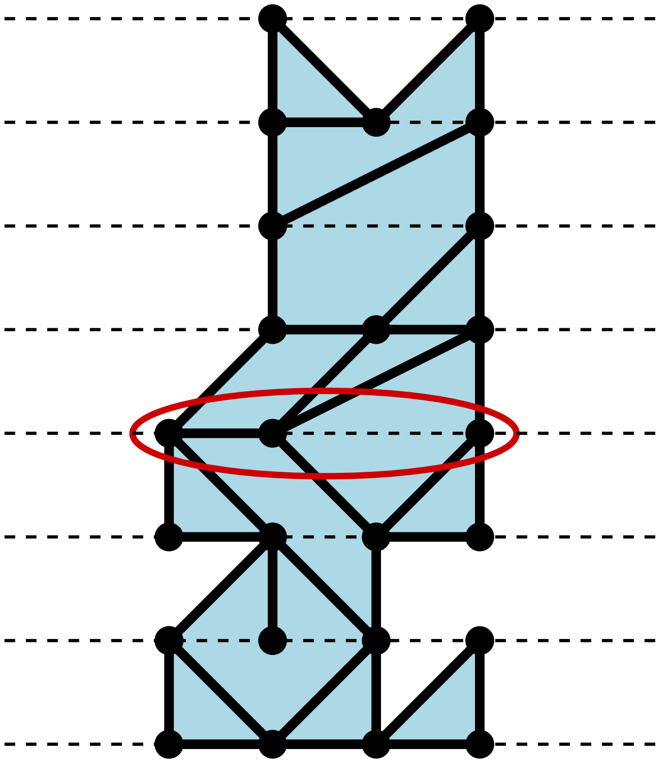


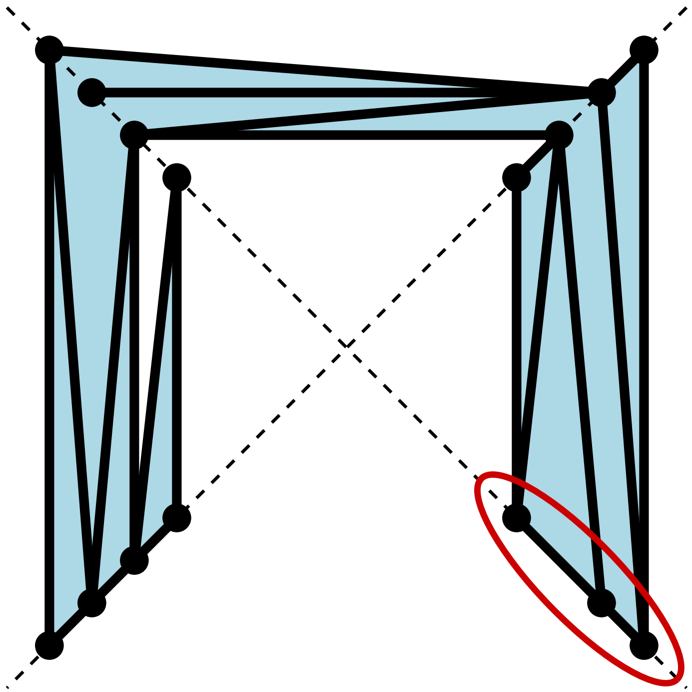


π_2^1 for Outerplanar Graphs

Outerplanar graphs are track drawable.

[Felsner, Liotta, Wismath, JGAA 2003]

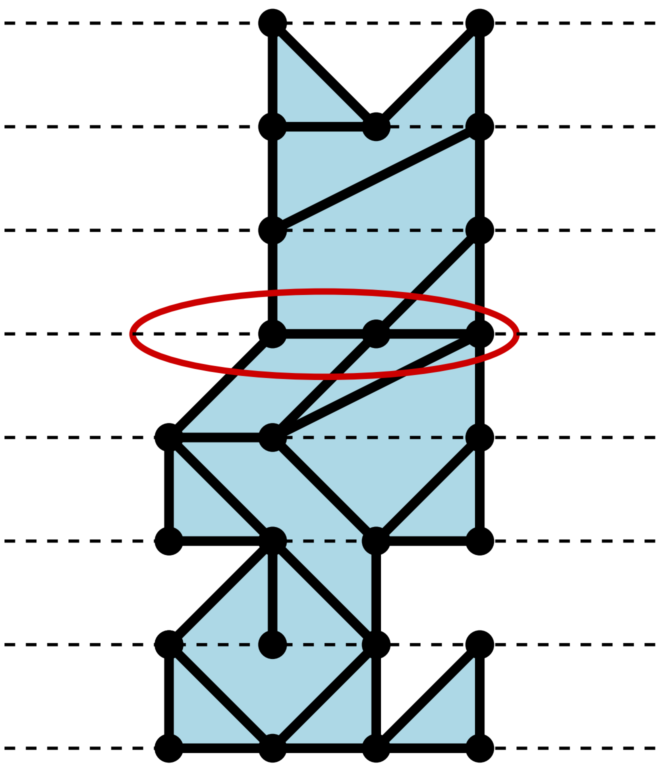


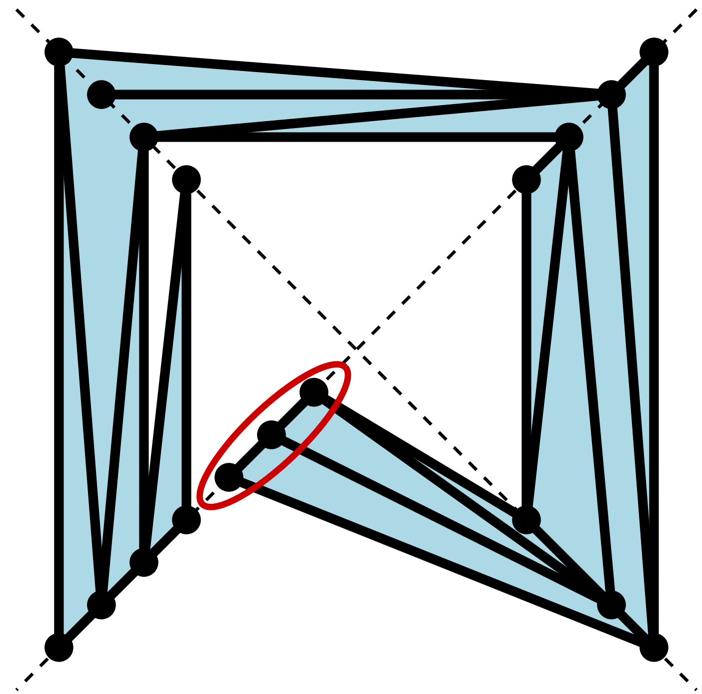


π_2^1 for Outerplanar Graphs

Outerplanar graphs are track drawable.

[Felsner, Liotta, Wismath, JGAA 2003]

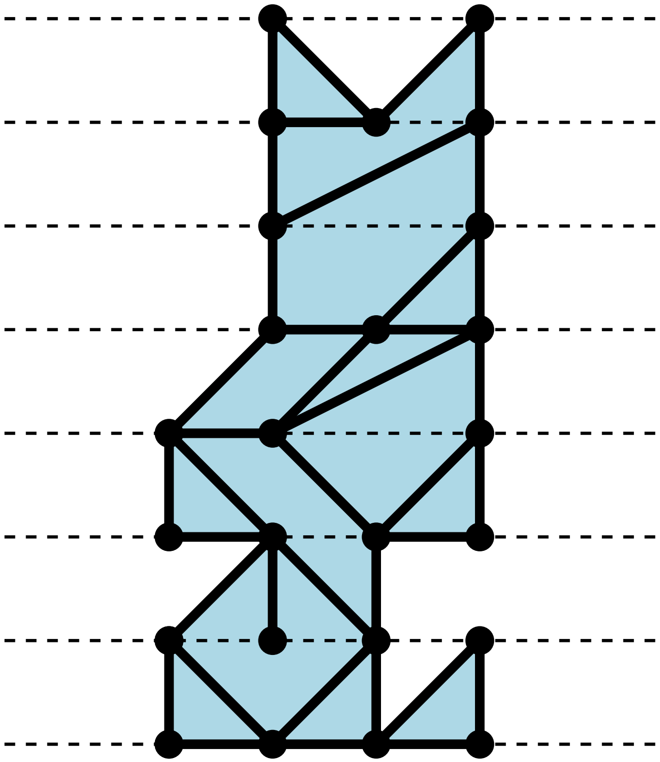


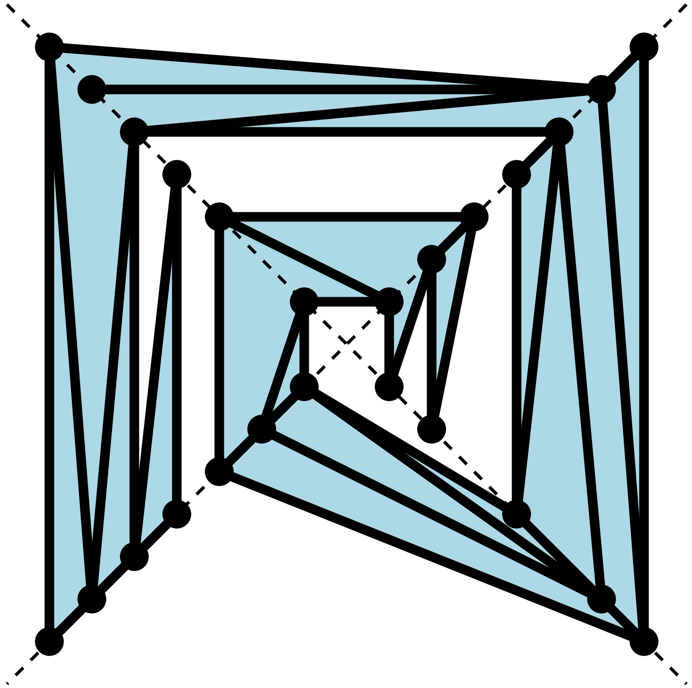


π_2^1 for Outerplanar Graphs

Outerplanar graphs are track drawable.

[Felsner, Liotta, Wismath, JGAA 2003]



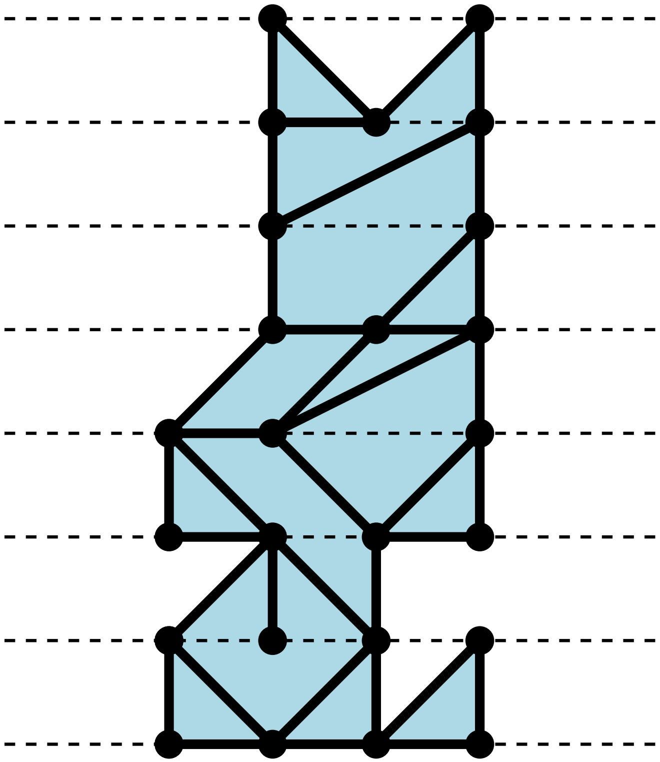


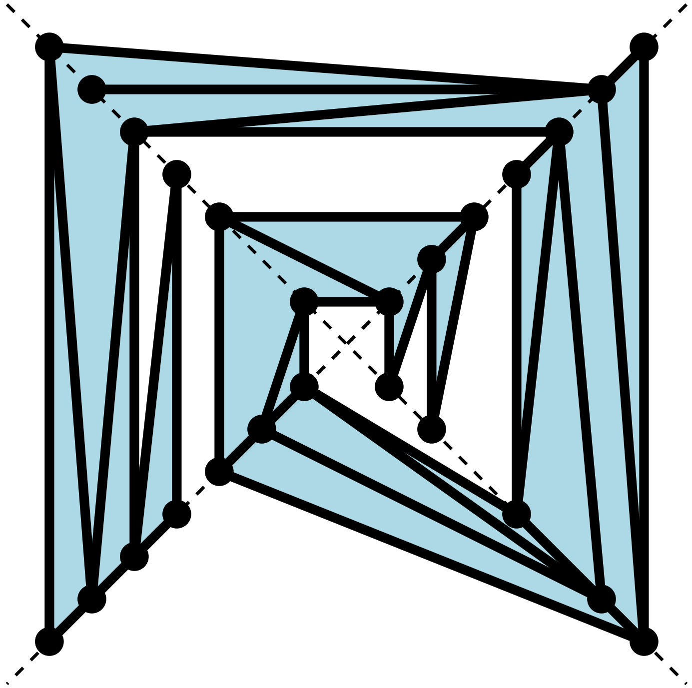
similar to result by Bannister et al. [GD 2016]

π_2^1 for Outerplanar Graphs

Outerplanar graphs are track drawable.

[Felsner, Liotta, Wismath, JGAA 2003]





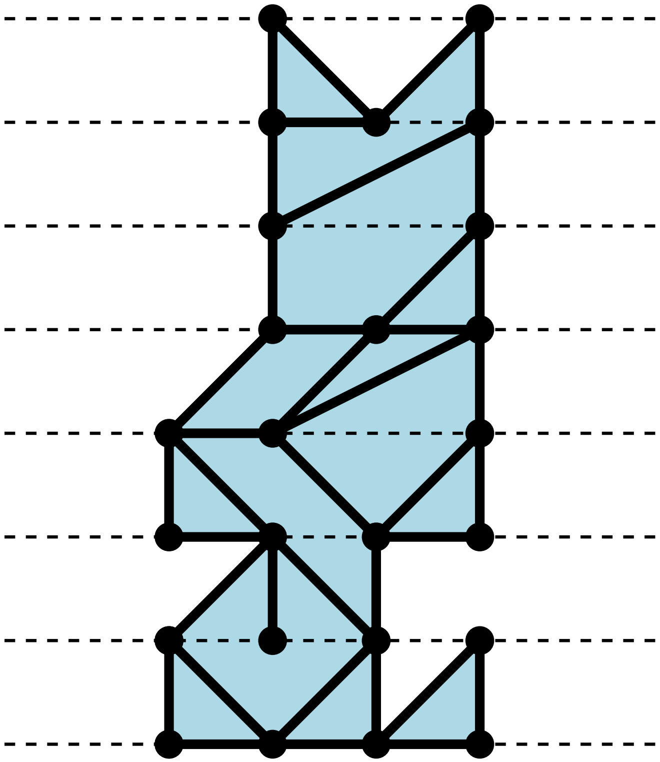
similar to result by Bannister et al. [GD 2016]

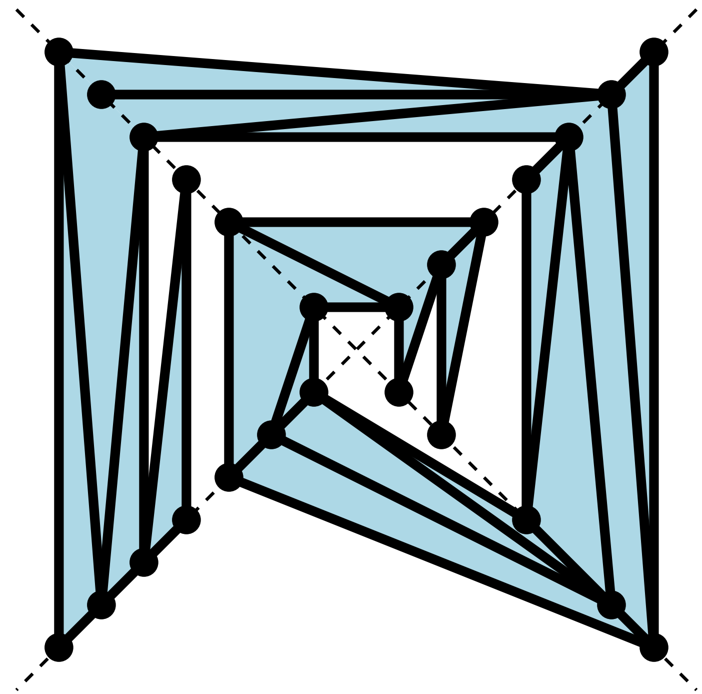
For any outerplanar graph G: $\pi_2^1(G) \leq 2$

π^1_2 for Outerplanar Graphs

Outerplanar graphs are track drawable.

[Felsner, Liotta, Wismath, JGAA 2003]





similar to result by Bannister et al. [GD 2016]

For any outerplanar graph G: $\pi^1_2(G) \leq 2$

On the other hand: infinitely many triangulations G with $\Delta(G) \leq 12$ and $\pi^1_2(G) \geq n^{0.01}$

Further Problems that We Considered

- Variants with parallel lines: $\bar{\pi}_2^1$ and $\bar{\pi}_3^1$

Further Problems that We Considered

- Variants with parallel lines: $\bar{\pi}_2^1$ and $\bar{\pi}_3^1$
- Topological variant — don't insist on straight lines: $\tau_3^2(G) = \text{thickness}(G)$

Further Problems that We Considered

- Variants with parallel lines: $\bar{\pi}_2^1$ and $\bar{\pi}_3^1$
- Topological variant — don't insist on straight lines: $\tau_3^2(G) = \text{thickness}(G)$
- Exact values for some of our parameters for complete bipartite graphs

Further Problems that We Considered

- Variants with parallel lines: $\bar{\pi}_2^1$ and $\bar{\pi}_3^1$
- Topological variant — don't insist on straight lines: $\tau_3^2(G) = \text{thickness}(G)$
- Exact values for some of our parameters for complete bipartite graphs
- Separation between ρ_2^1 and ρ_3^1: there are planar graphs G with $\rho_2^1(G) = \Omega(n)$ and $\rho_3^1(G) = O(n^{2/3})$

Open problems

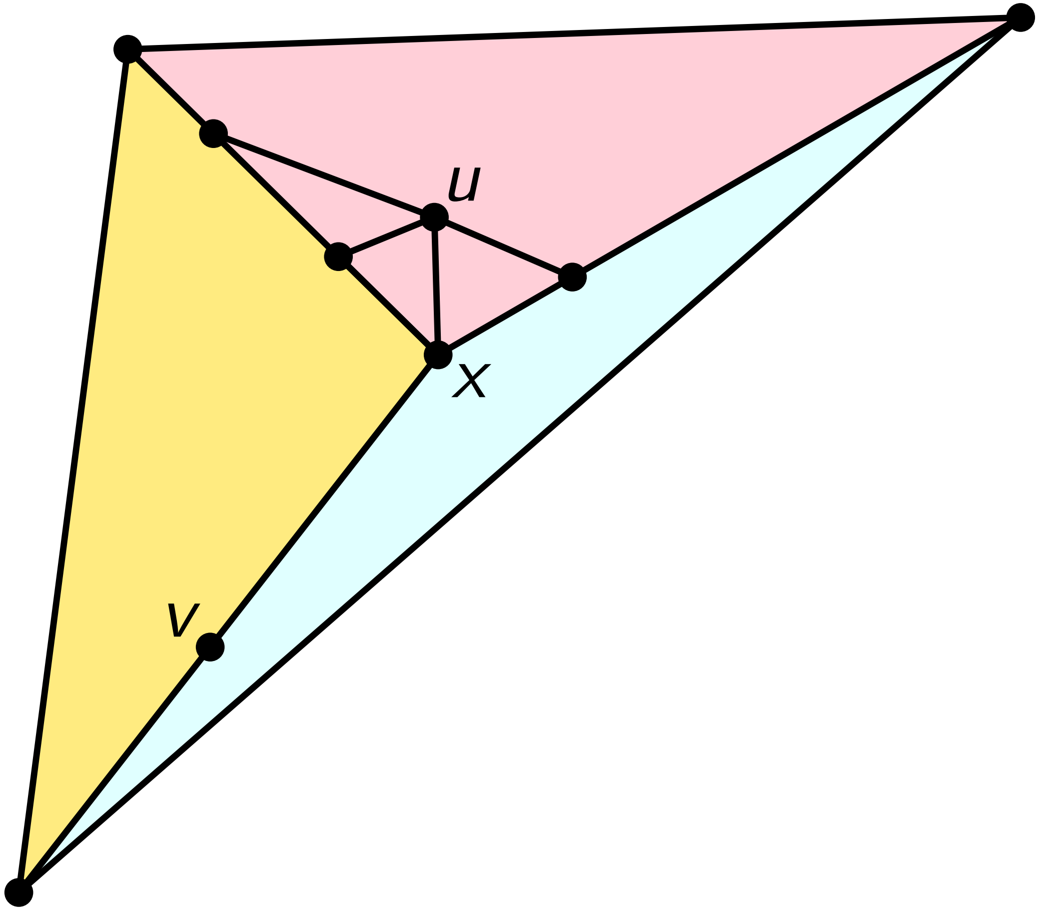
Near-planar graphs

Task: add one edge uv to a planar graph G and create a ρ_3^2-optimal drawing.

Near-planar graphs

Task: add one edge uv to a planar graph G and create a ρ_3^2-optimal drawing.

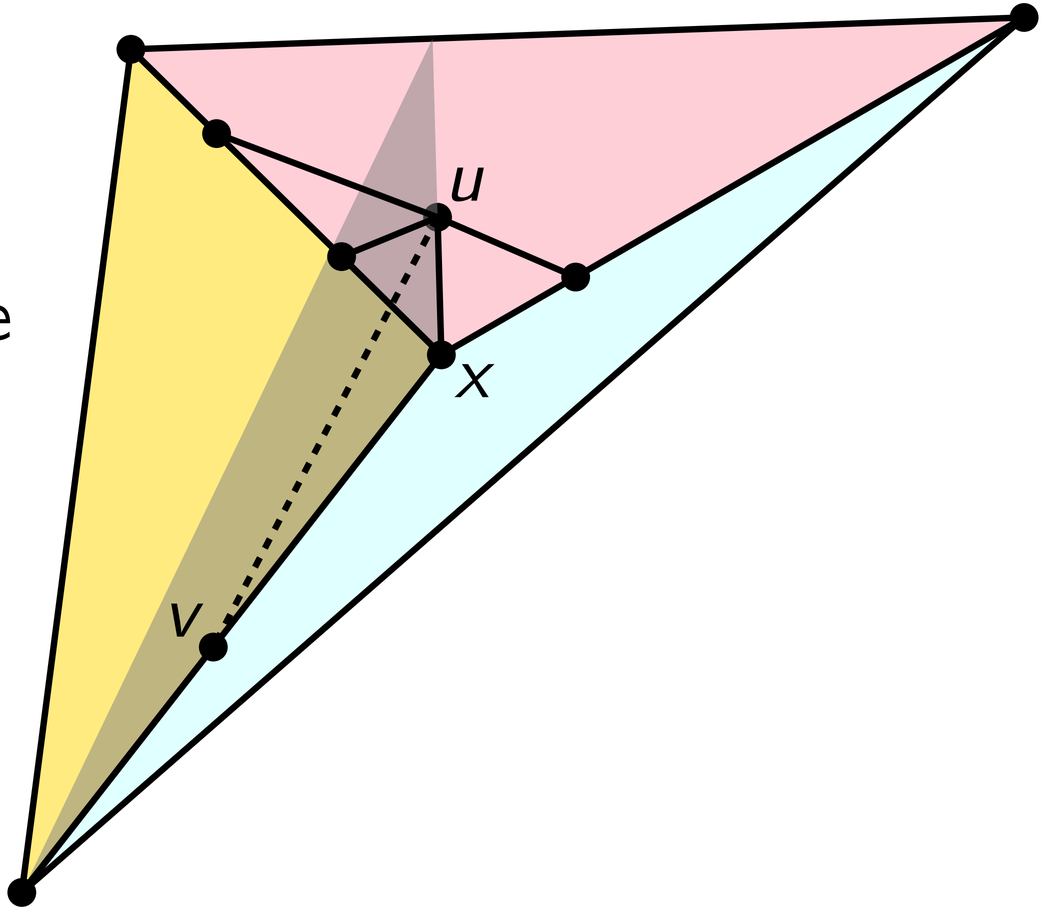
- Split the planar graph onto three planes



Near-planar graphs

Task: add one edge uv to a planar graph G and create a ρ_3^2-optimal drawing.

- Split the planar graph onto three planes
- Add uv on a fourth plane

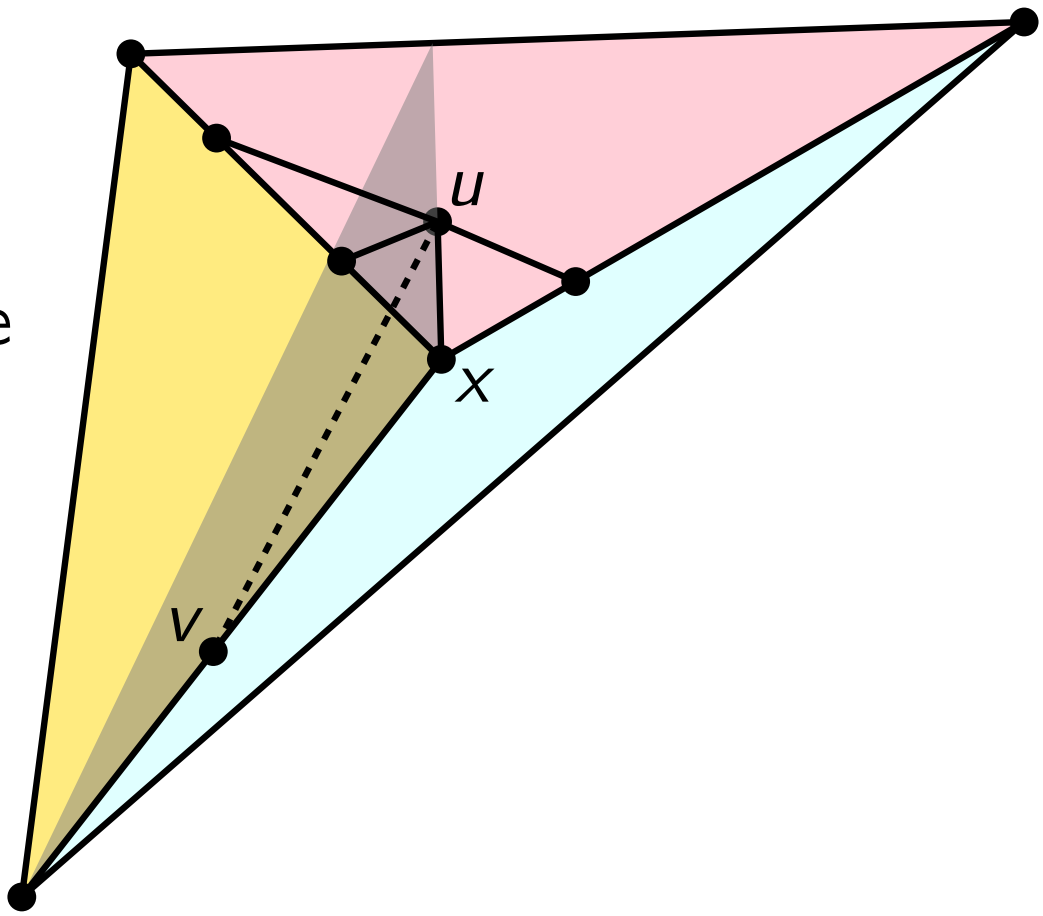


Near-planar graphs

Task: add one edge uv to a planar graph G and create a ρ_3^2-optimal drawing.

- Split the planar graph onto three planes
- Add uv on a fourth plane

$\Rightarrow \rho_3^2(G + uv) \leq 4$

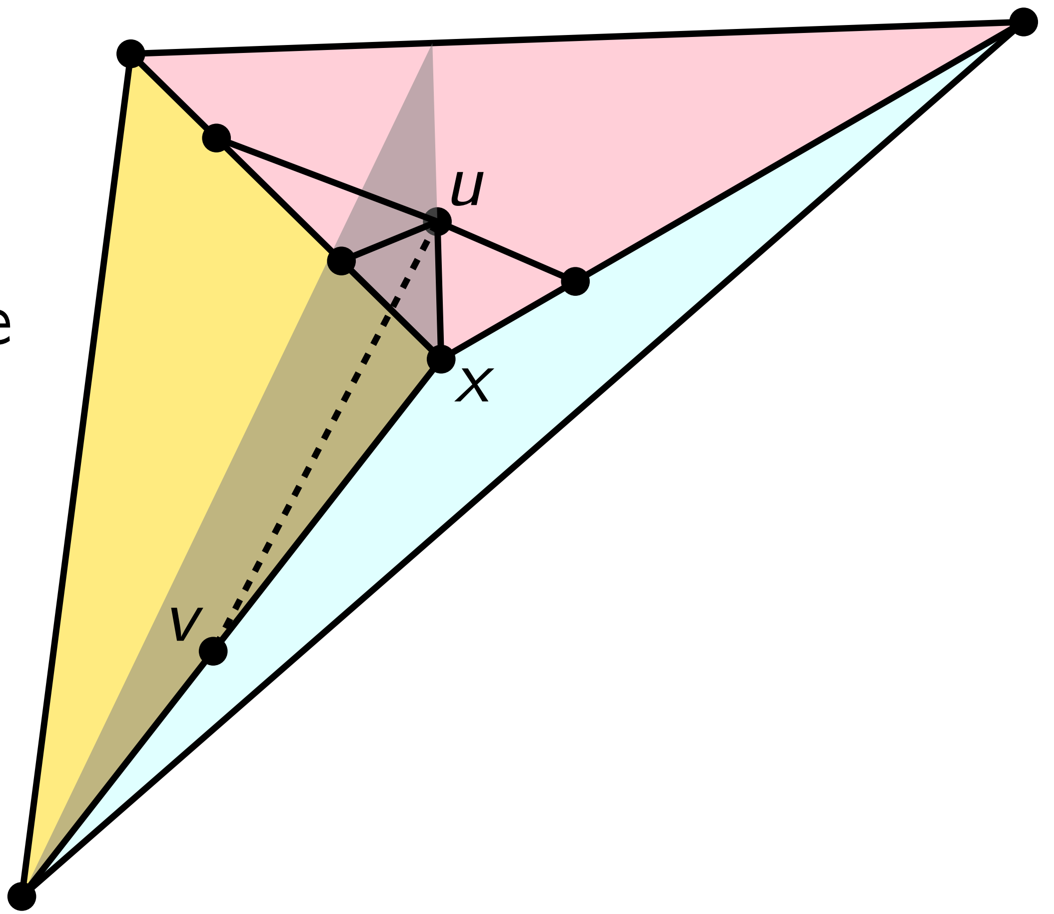


Near-planar graphs

Task: add one edge uv to a planar graph G and create a ρ_3^2-optimal drawing.

- Split the planar graph onto three planes
- Add uv on a fourth plane

$\Rightarrow \rho_3^2(G + uv) \leq 4$



Is this bound worst-case optimal?

Open Problems

- We know $n^2/12 \lessapprox \rho_3^2(K_n) \lessapprox n^2/6$.

Open Problems

- We know $n^2/12 \lessapprox \rho_3^2(K_n) \lessapprox n^2/6$.
  Determine exact values!

n	4	5	6	7	8	9	>10
$\rho_3^2(K_n) \geq$	1	3	4	6	6	7	**?**
$\rho_3^2(K_n) \leq$	1	3	4	6	7	**?**	**?**

Open Problems

- We know $n^2/12 \lessapprox \rho_3^2(K_n) \lessapprox n^2/6$.
  Determine exact values!

n	4	5	6	7	8	9	>10
$\rho_3^2(K_n) \geq$	1	3	4	6	6	7	**?**
$\rho_3^2(K_n) \leq$	1	3	4	6	7	**?**	**?**

- Is $\pi_2^1(G) = o(n)$ for all planar graphs?
  Known: $\pi_3^1(G) \leq 3$ for planar graphs
  and $\pi_2^1(G) \leq 2$ for outerplanar graphs

Open Problems

- We know $n^2/12 \lessapprox \rho_3^2(K_n) \lessapprox n^2/6$.
  Determine exact values!

n	4	5	6	7	8	9	>10
$\rho_3^2(K_n) \geq$	1	3	4	6	6	7	**?**
$\rho_3^2(K_n) \leq$	1	3	4	6	7	**?**	**?**

- Is $\pi_2^1(G) = o(n)$ for all planar graphs?
  Known: $\pi_3^1(G) \leq 3$ for planar graphs
  and $\pi_2^1(G) \leq 2$ for outerplanar graphs

- Is $\rho_3^1(G) = \rho_2^1(G)$ for trees?

Open Problems

- We know $n^2/12 \lessapprox \rho_3^2(K_n) \lessapprox n^2/6$.
  Determine exact values!

n	4	5	6	7	8	9	>10
$\rho_3^2(K_n) \geq$	1	3	4	6	6	7	**?**
$\rho_3^2(K_n) \leq$	1	3	4	6	7	**?**	**?**

- Is $\pi_2^1(G) = o(n)$ for all planar graphs?
  Known: $\pi_3^1(G) \leq 3$ for planar graphs
  and $\pi_2^1(G) \leq 2$ for outerplanar graphs

- Is $\rho_3^1(G) = \rho_2^1(G)$ for trees?

- Bound ρ_3^2 for 1-planar graphs or RAC graphs.