Low Ply Drawings of Trees

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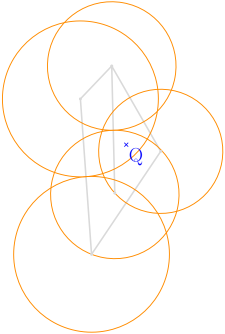
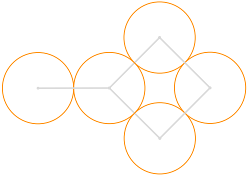
Ply number of graph (drawing)

Let G be a graph and Γ be a straight-line drawing of G.

- To any vertex v of Γ we assign an open ply-disc D_v centered at v with radius r_v equal to the half of the length of the longest edge incident to v.
- For any point $Q \in \mathbb{R}^2$, denote by S_Q the set of ply-discs that contain Q.
- The ply-number of Γ is $\text{pn}(\Gamma) = \max_{Q \in \mathbb{R}^2} |S_Q|$.
- Set the ply-number of graph G as

$$\text{pn}(G) = \min_{\Gamma(G)} \text{pn}(\Gamma) = \min_{\Gamma(G)} \max_{Q \in \mathbb{R}^2} |S_Q|.$$





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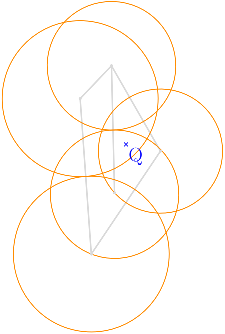

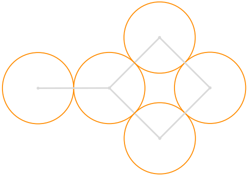

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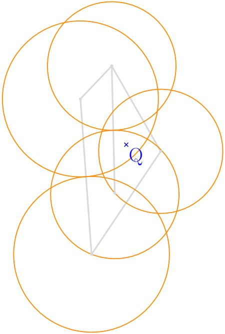

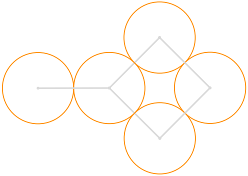

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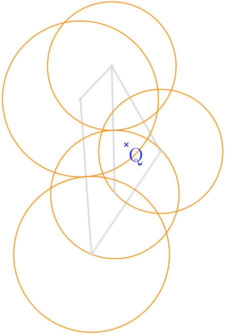

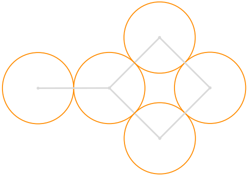

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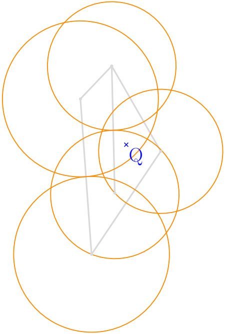

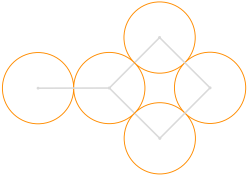

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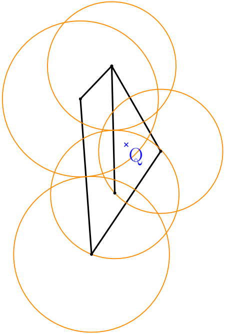

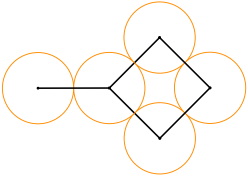

Motivation and Inspiration

- New aesthetic criterion - Graph drawings with small ply-number distribute vertices uniformly.
- Road networks - Eppstein and Goodrich analyzed real-world road networks from the point of view of ply of the geometric layout.
- Spheres of influence - Real geographic networks usually have constant-ply.
- Small stress of graph layout - Low ply drawings seems to minimize the stress of a drawing, measured by the weighted sum $\sum_{i,j \in V} w_{i,j}(||p_i - p_j|| - d_{i,j})^2$ of differences between the Euclidean distance and graph-theoretic distance.

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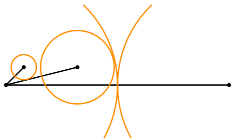
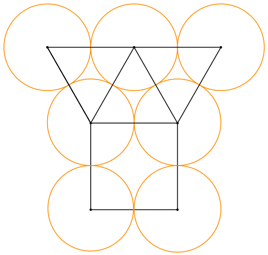
Facts about ply-numbers





Area of a drawing Γ is an area of the smallest axis-aligned rectangle containing the drawing, under the resolution rule that each edge has length at least 1.

What is already known:

- Graphs with ply-number 1 are exactly the graphs that have circle contact representation with unit circles.
- It is NP hard to test wheather G has ply-number 1.
- Binary trees, stars and caterpillars have ply-number 2.
- Trees with depth h have drawings with ply-number $h + 1$.

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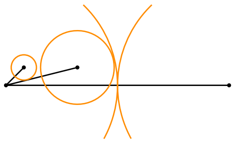

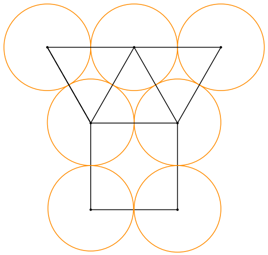

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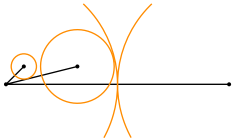

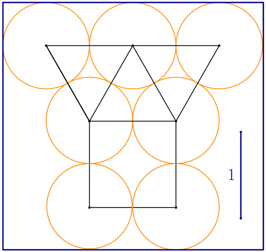

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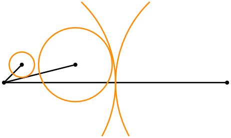

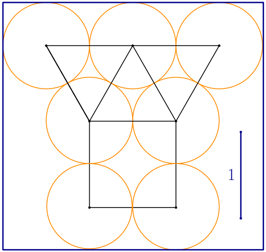

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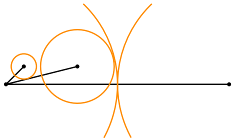

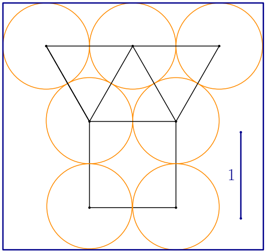

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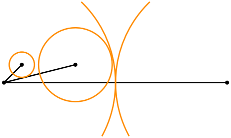

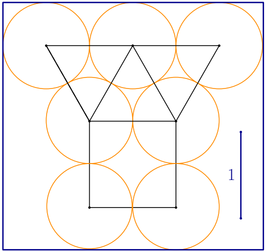

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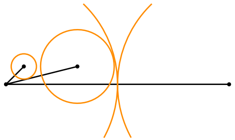

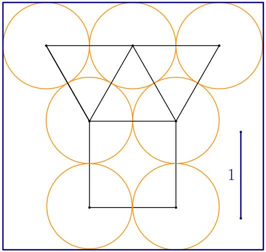

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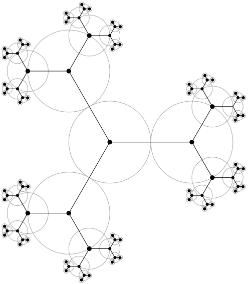
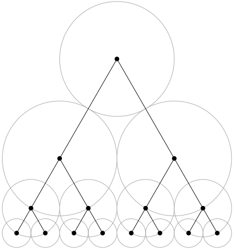
Questions that awaits answers





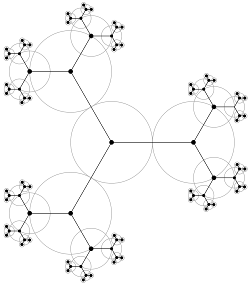
Arising questions:

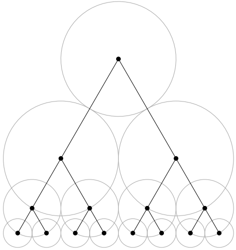
- Is it possible to draw a binary tree, a star, or a caterpillar in polynomial area with ply-number 2?
- Is it possible to draw ternary trees with ply-number 2?
- Do k-ary trees, for $k > 2$, have constant ply-number?
- Is there a correlation between the number of edge crossings and the ply-number? Or realtion between the stress of the graph and the ply-number?

Poster (Felice at al.)

An experimental Study on the Ply Number of the Straight-line Drawing

Questions that awaits answers





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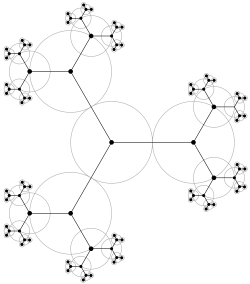

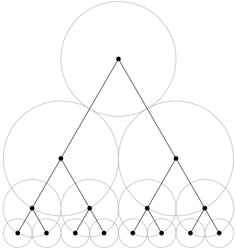

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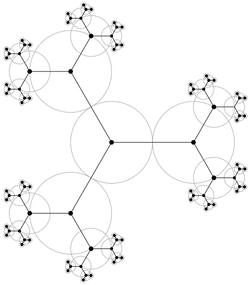

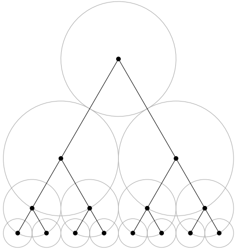

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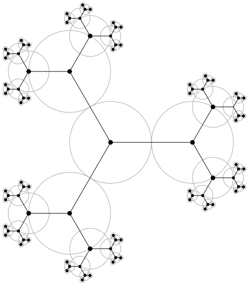

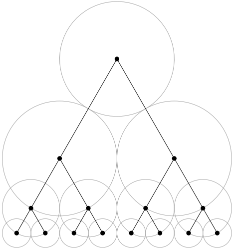

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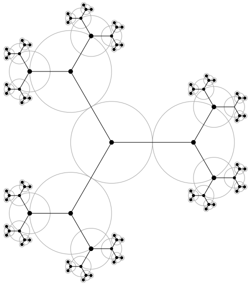

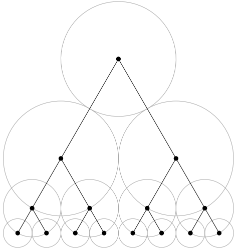

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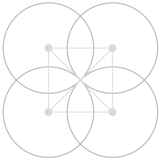
An experimental Study on the Ply Number of the Straight-line Drawing

Main results

We answer several of these questions.

Theorem

Any constant-ply drawing of star $K_{1,n-1}$ has exponential area.



Let T_{10}^h be the 10-ary tree of height h.

Theorem

For any $M > 0$ there is an integer $h > 0$ such that

$$\mathrm{pn}(T_{10}^h) > M$$

And study deeper the ply-number of k-ary trees:

Theorem

Every n-vertex 5-ary tree has a drawing with ply-number $O(\log n)$ and polynomial area.

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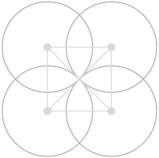

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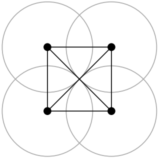

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We answer several of these questions.

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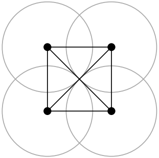

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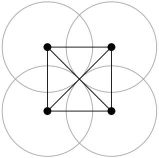

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Proof (by contradiction): We use a packing argument.

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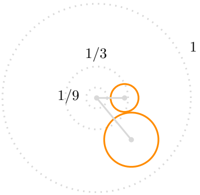

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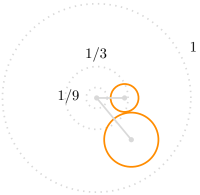

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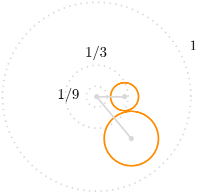

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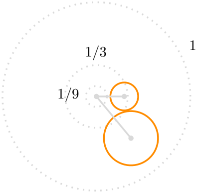

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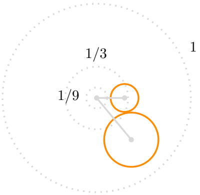

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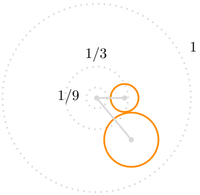

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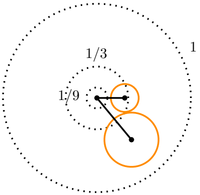

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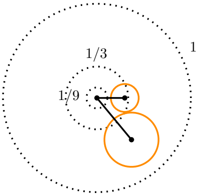

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Drawings of Bounded Degree Trees

Notation (for rooted trees):

- Edge e dominates edge f, write $e >_D f$, if both edges lie on the same branch of tree T_{10}^h and $\ell(e) \geq 3^{s+1}\ell(f)$, where s is the number of edges between e and f.
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Observation: Any chain $e_1 >_{FD} e_2 >_{FD} \cdots >_{FD} e_M$ of M edges in T_{10}^h guarantees $\mathrm{pn}(T_{10}^h) \geq M$.

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For any $M > 0$ there is an integer $h > 0$ such that $\mathrm{pn}(T_{10}^h) > M$.

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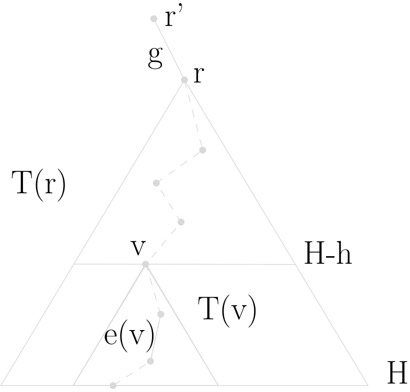

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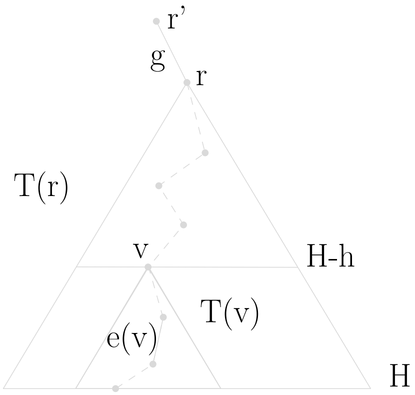

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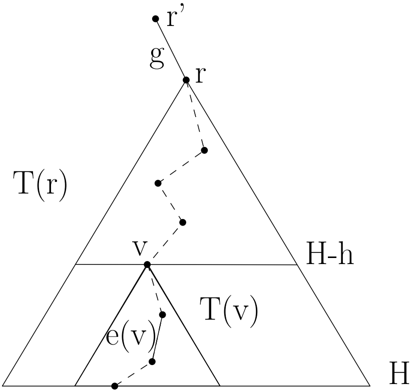

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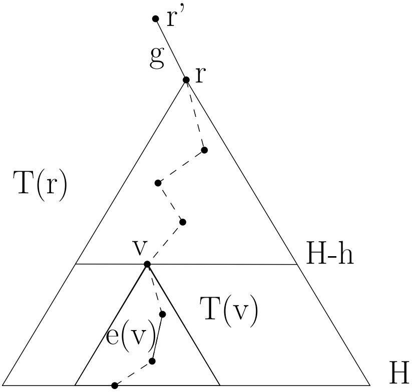

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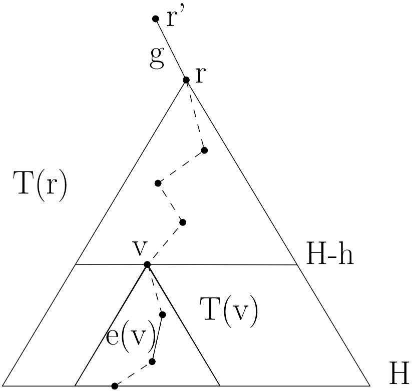

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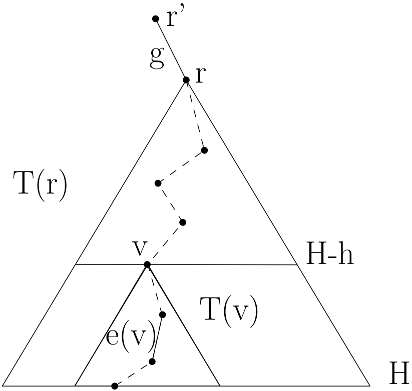

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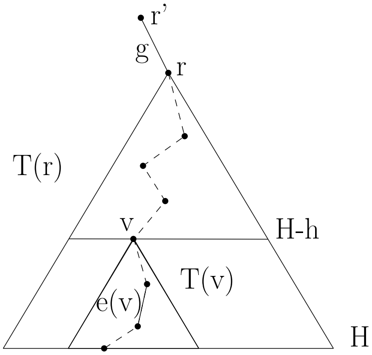

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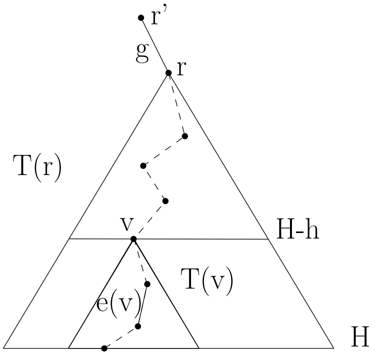

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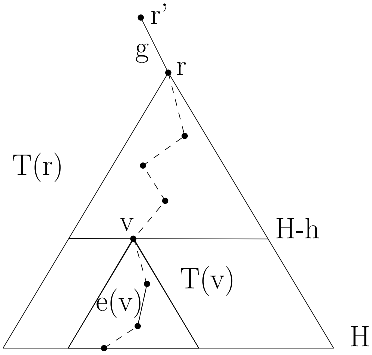

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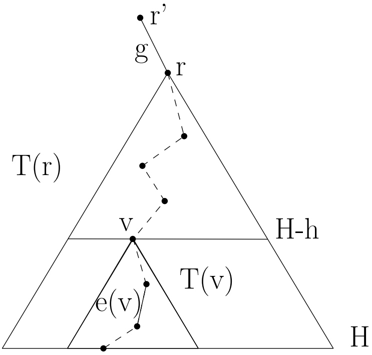

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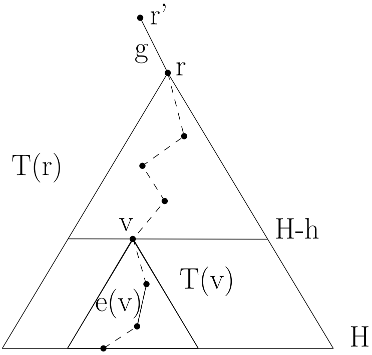

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$$\ell(e(v)) \in [3^k, 3^{k+1})$$

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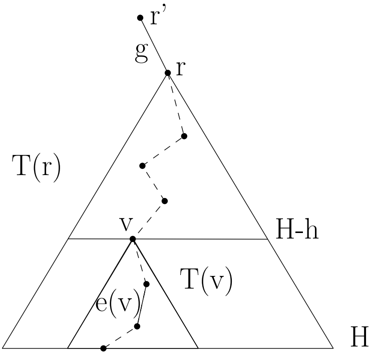

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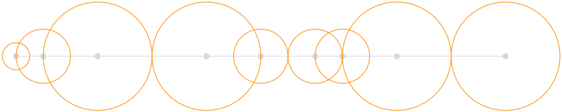
Log-ply Drawings of Trees

Theorem

Every n-vertex 5-ary tree has a drawing with ply-number $O(\log n)$ and $O(n^{7.2})$ area.

Proof (sketch for ternary trees):

- We decompose any ternary tree T into paths; that is, we construct a heavy-path tree $\mathcal{T}$ of T with depth $O(\log n)$.
- Then construct the drawing of these paths along one line with ply-number 2.



- Finally glue scaled paths in axis alligned manner.
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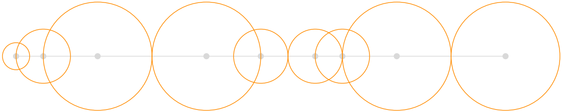

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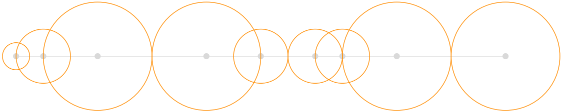

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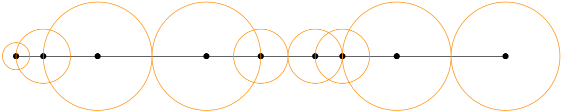

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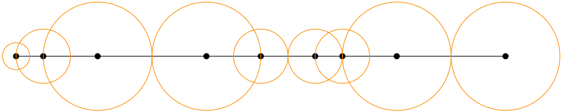

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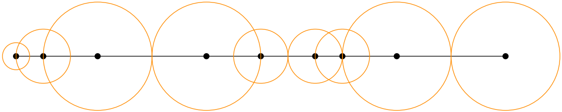

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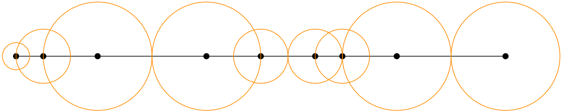

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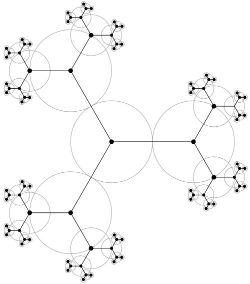

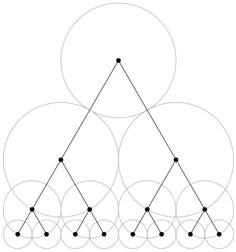

Although we answered two questions, we provide more than two new ones, that that the number of open questions does not decrease:

- Is it possible to draw a binary tree in polynomial area with ply-number 2?
- Binary trees admit constant-ply drawing, 10-ary trees do not admit such drawings. What about k-ary trees for $k \in \{3, 4, \ldots, 9\}$?
- What types of trees (or classes of graphs) always admit log-ply drawing?
- Can you describe the correlation between the stress of the graph and the ply-number?
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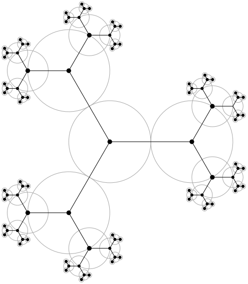

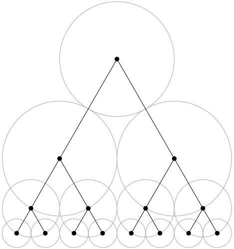

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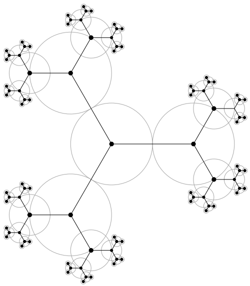

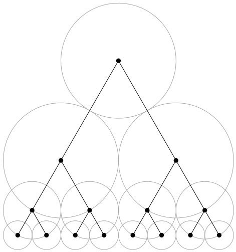

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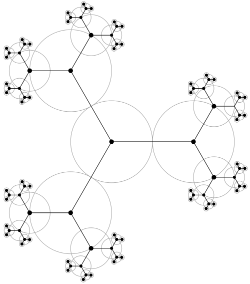

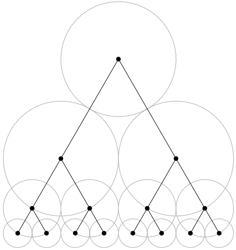

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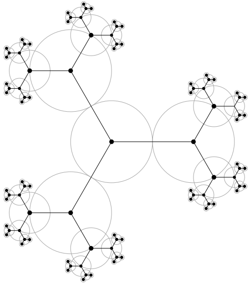

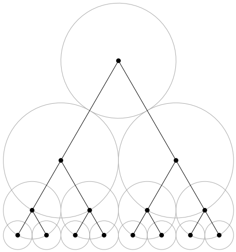

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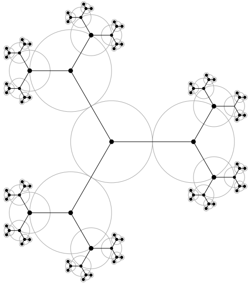

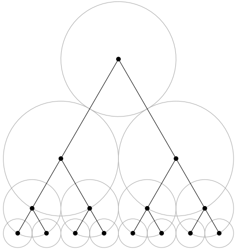

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