The crossing number of the cone of a graph

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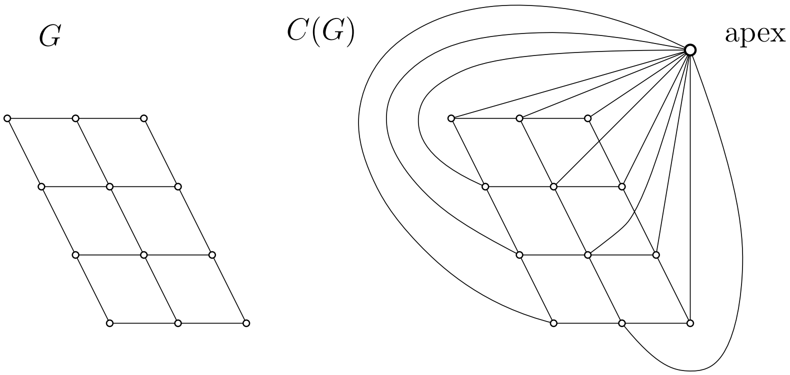
Cone of a graph

Cone of G: $C(G)$ . . . add a universal vertex (apex)



Motivation: Two conjectures and the cone

- Harary-Hill Conjecture:

$$cr(K_n) = \begin{cases} \frac{1}{64} n(n-2)^2(n-4), & n \text{ is even;} \\ \frac{1}{64}(n-1)^2(n-3)^2, & n \text{ is odd.} \end{cases}$$

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[1] In the H-H Conjecture: $K_{n+1} = C(K_n)$.

[2] In a special case for Albertson's conjecture, $\chi(C(G)) = \chi(G) + 1$.

Motivation: Two conjectures and their variation

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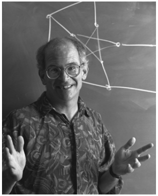 

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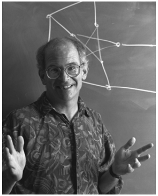

A variation asked by Bruce Richter: Given $n \geq 5$ and a graph G with $cr(G) \geq cr(K_n)$, does it follow that $cr(CG) \geq cr(K_{n+1})$?

Richter's question

$$cr(G) \geq cr(K_n) \;\Rightarrow\; cr(CG) \geq cr(K_{n+1})?$$

Observation: True for $n = 5$.

(Because $C(K_5) = K_6$ and $cr(C(K_{3,3})) = 3$)

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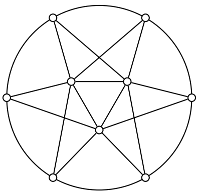
False for $n = 6$.



$cr(G) = 3 = cr(K_6)$ and $cr(C(G)) = 6 < cr(K_7) = 9$

New question

Problem: Suppose $cr(G) = k$. How much bigger is $cr(C(G))$?

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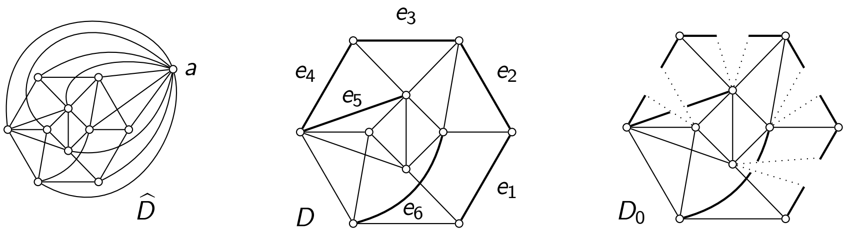
Theorem. $\boxed{\sqrt{k/2} \leq \phi(k) \leq \sqrt{3k}}$

Upper bound: Previous graph with edge-multiplicities r has crossing number $3r^2$ and its cone has crossing number $3r^2 + 3r$.

The lower bound

Theorem. $cr(C(G)) \geq cr(G) + \sqrt{\frac{1}{2} cr(G)}$

Proof. Step 1: Obtain a 1-page drawing of G:



(a) (b) (c)

The lower bound

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Proof (cont'd). From a 1-page drawing to a 2-page drawing.

Lemma (Edwards 1975)

G graph of order n with $m \geq 1$ edges. Then G contains a bipartite subgraph with at least $\frac{1}{2}m + \sqrt{\frac{1}{8}m + \frac{1}{64}} - \frac{1}{8} > \frac{1}{2}m$ edges.

Corollary

Let D be a 1-page drawing of a graph G with $k \geq 1$ crossings. Then some edges of G can be redrawn in a new page, obtaining a 2-page drawing with $\leq \frac{1}{2}k - \sqrt{\frac{1}{8}k + \frac{1}{64}} + \frac{1}{8}$ crossings.
Such a drawing can be found in time $O(|E(G)| + k)$ (Bollobás & Scott, 2002).

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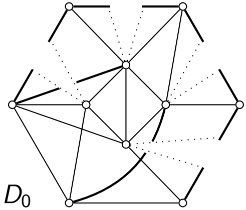
Proof (cont'd).



If too few crossings, we obtain a drawing of G with $< cr(G)$ crossings. □

Simple graphs

$\phi_s(k) := \min\{cr(C(G)) - cr(G) \mid cr(G) = k, G \text{ is simple}\}$

The lower bound $\phi_s(k) \geq \sqrt{k/2}$ still holds.

Theorem. $\phi_s(3) = 3,\ \phi_s(4) = 4,\ \phi_s(5) = 5.$

Obvious conjecture!

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Theorem. $\boxed{\phi_s(k) = O(k^{3/4})}$

Conjecture

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This specific form of the conjecture is due to the following observation:

Proposition

If the Harary-Hill conjecture holds, then

$$\phi_s(k) \leq \sqrt{2}\, k^{3/4}(1 + o(1))$$

Note: This is indeed very close to the original question of Bruce Richter.

Questions?