19-21 September 2016
Athens, Greece

# On the density of non-Simple 3-Planar Graphs

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# Beyond Planarity

## Restrictions:

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- number of crossings

$k$-planar

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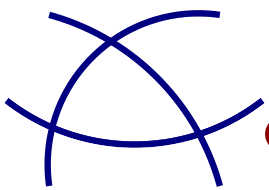
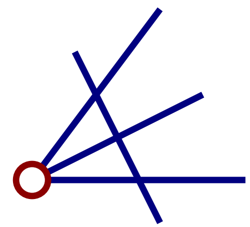


quasi-planar



fan-planar
fan-crossing free

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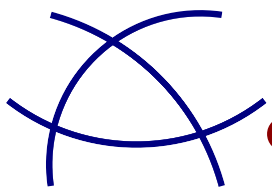

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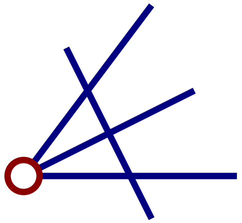

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- crossing angle

  RAC, LAC

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## 1-planar graphs

- optimal 1-planar graphs have $4n - 8$ edges
  characterization
- maximal 1-planar graphs have at least $2.22n$ edges
  [*Barát, Tóth* 15]
- There exist maximal 1-planar graphs
  with $2.647n$ edges [*Brandenburg et al.* 13]

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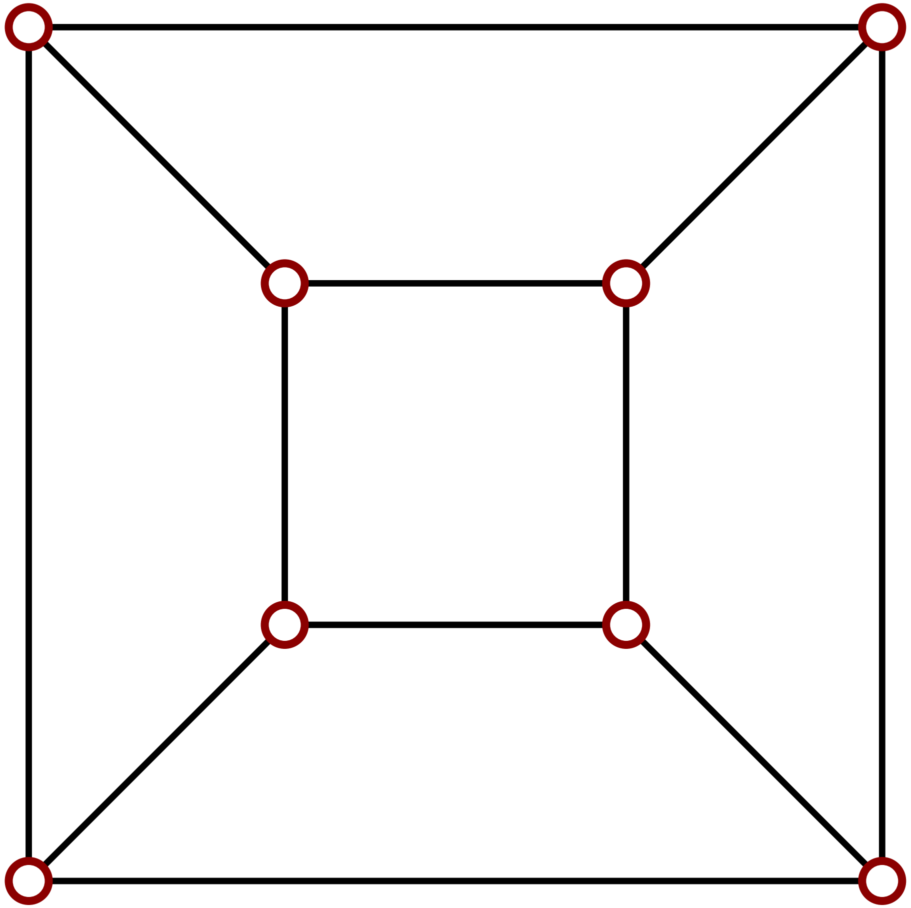

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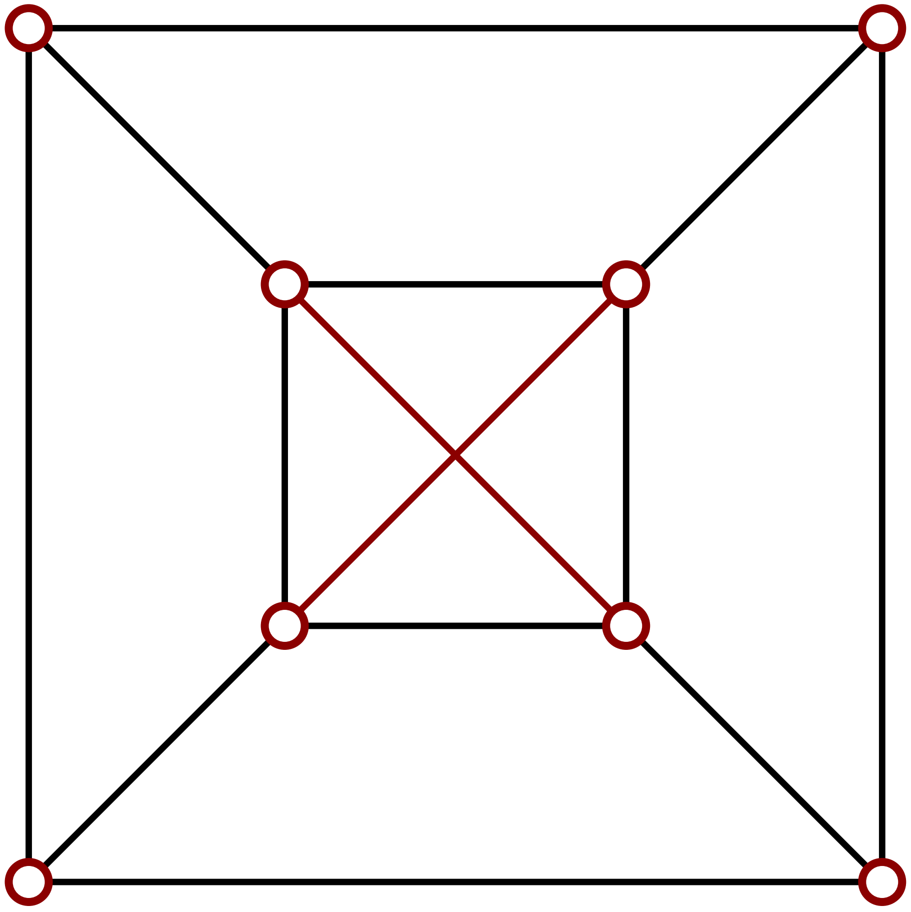

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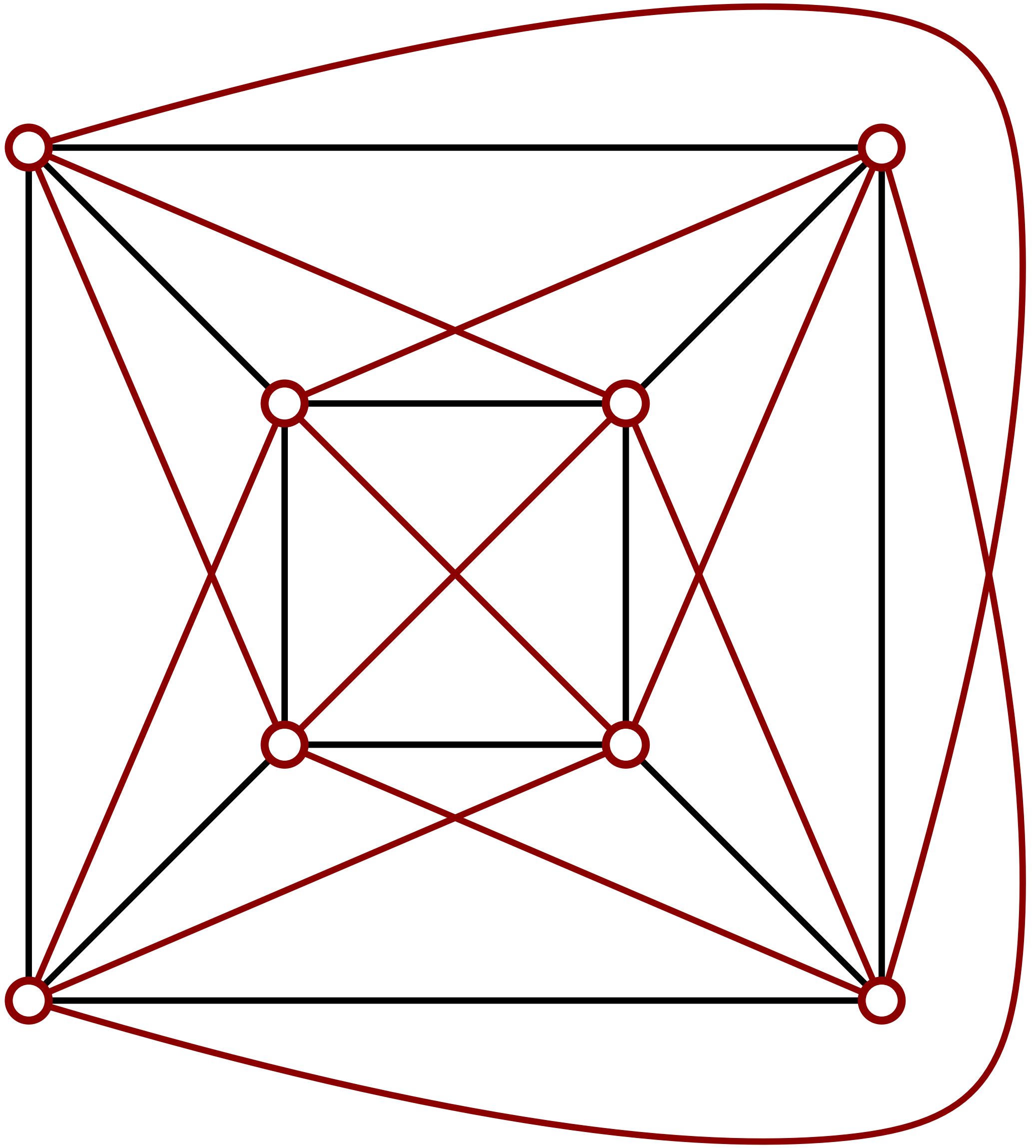

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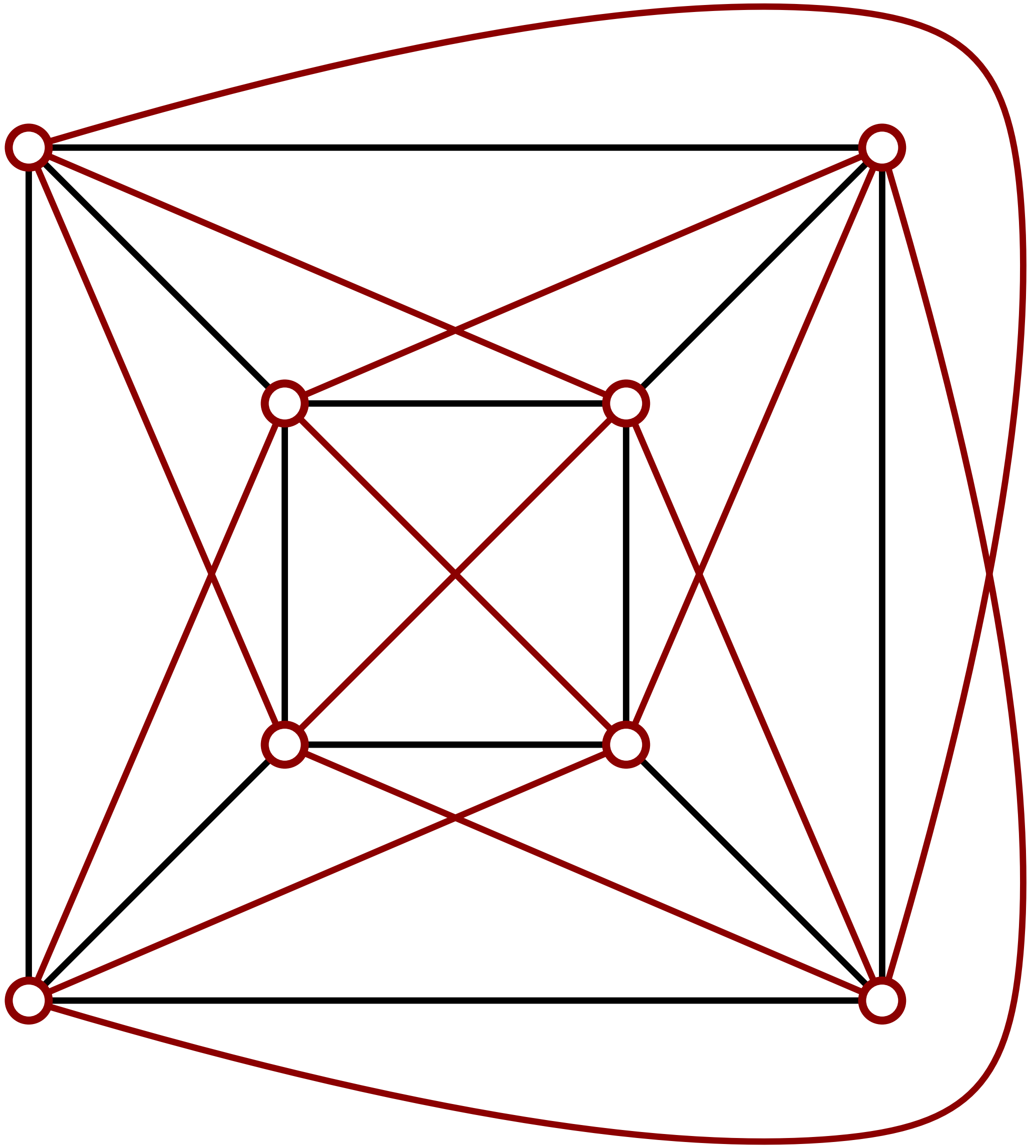

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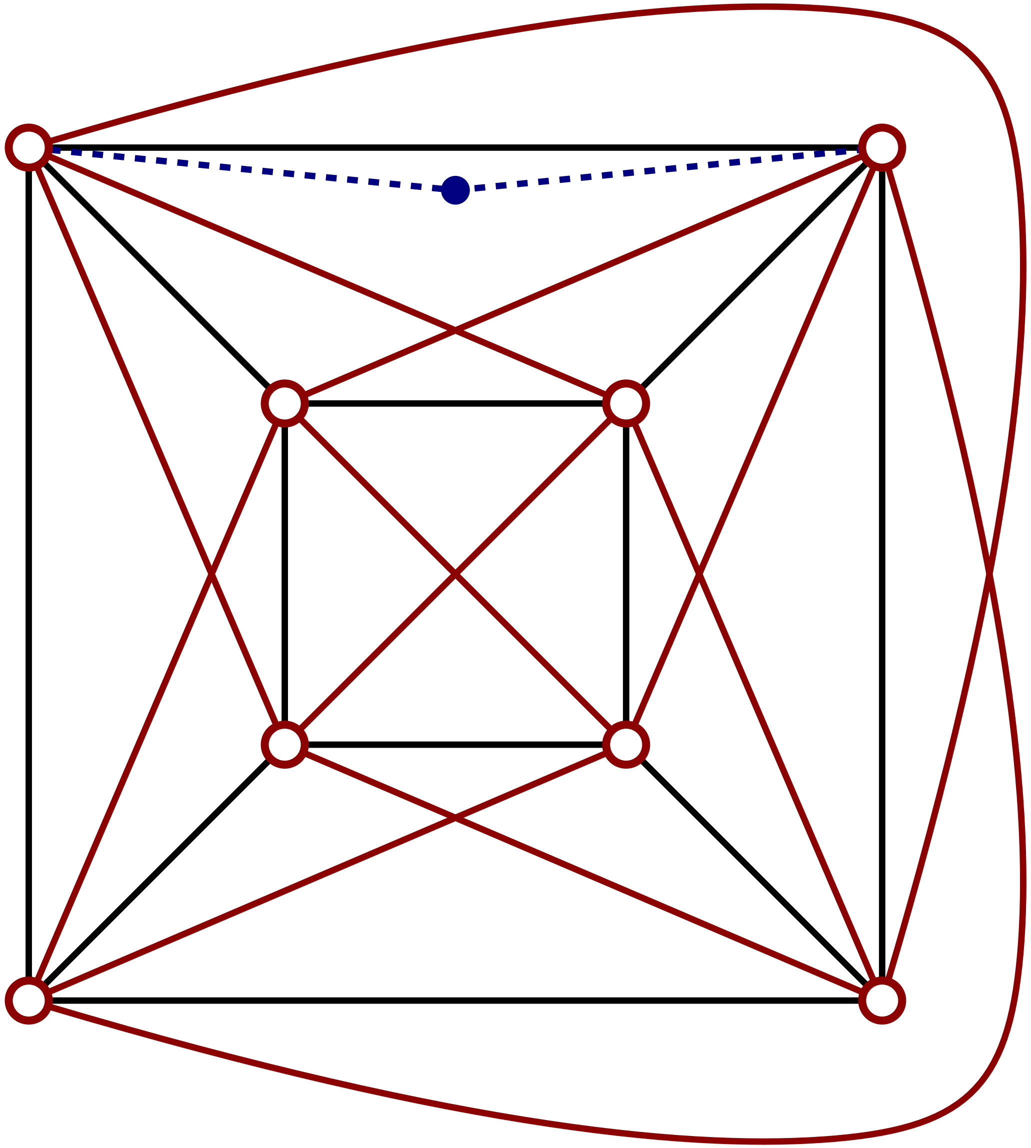

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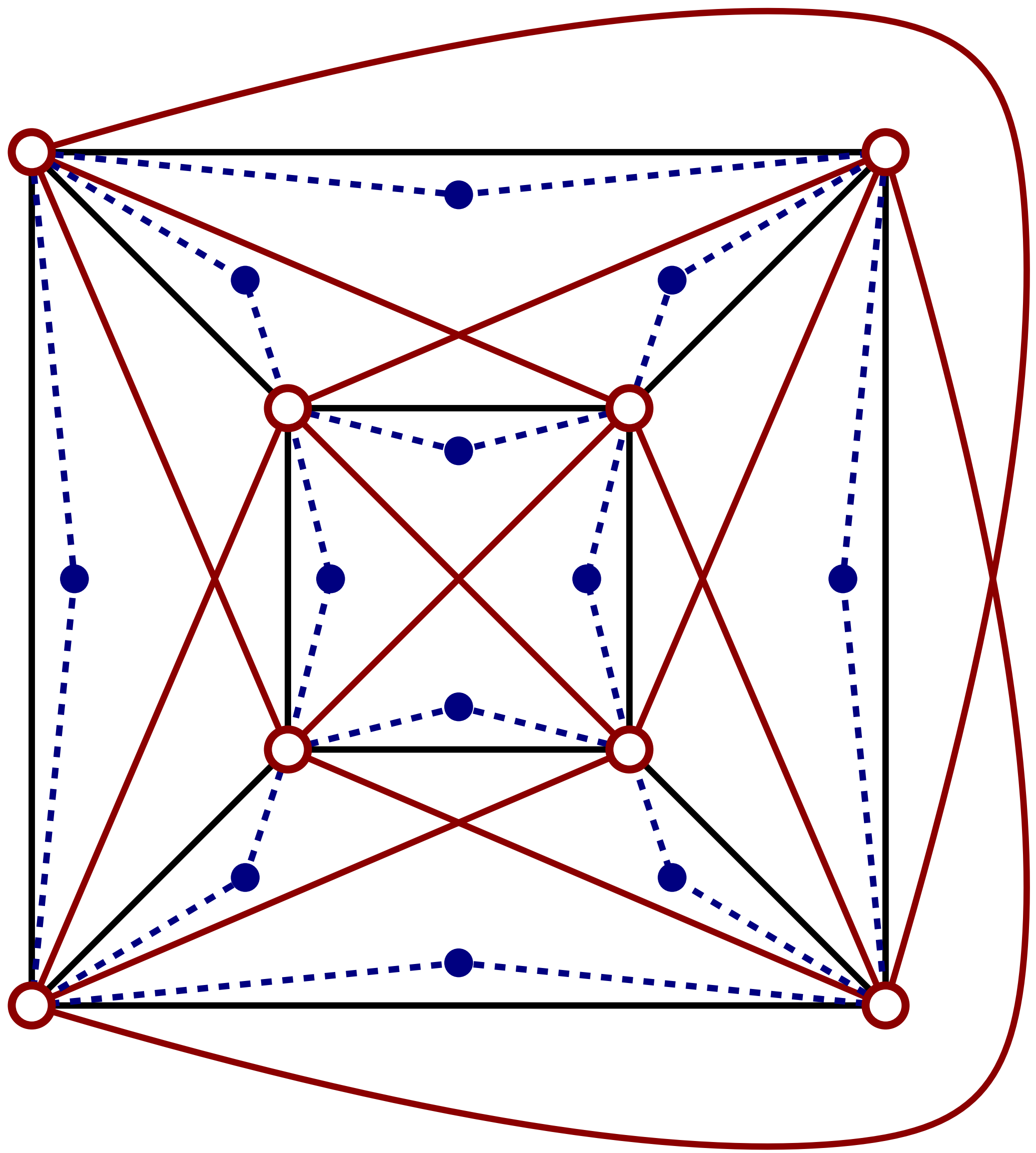

# Density of k-planar graphs

| | Upper Bound | |
|---|---|---|
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| 3-planar | $6n - 12$ | |
| 4-planar | $7n - 14$ | |

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$k \leq 2 \rightarrow$ **Tight**

$k \geq 3 \rightarrow$ **?**

# Our Contribution

**non-simple 3-planar graphs:**

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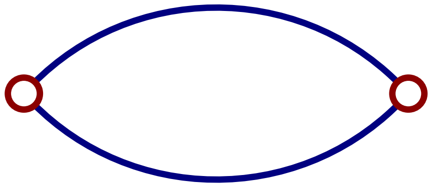
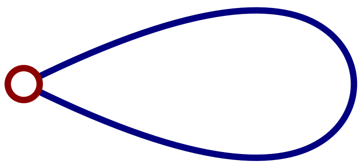
## non-simple 3-planar graphs:

Allow *non-homotopic* parallel edges or loops





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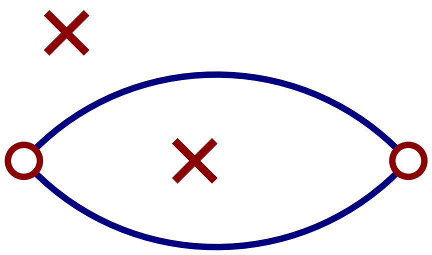

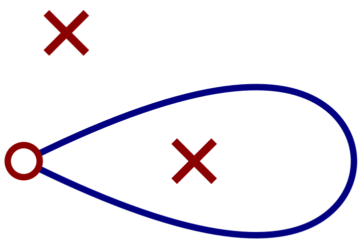

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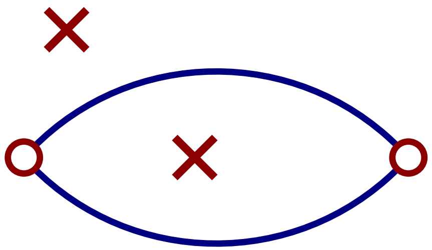

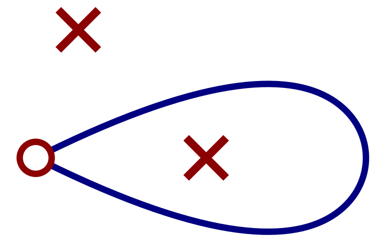

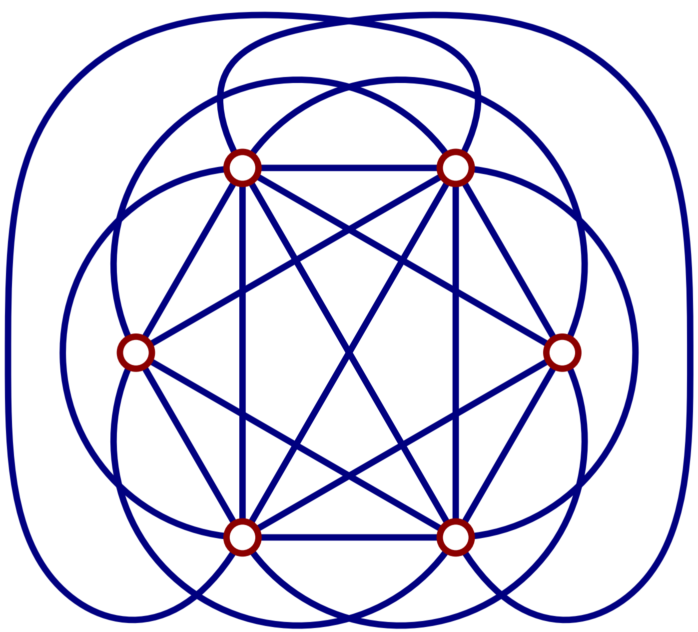

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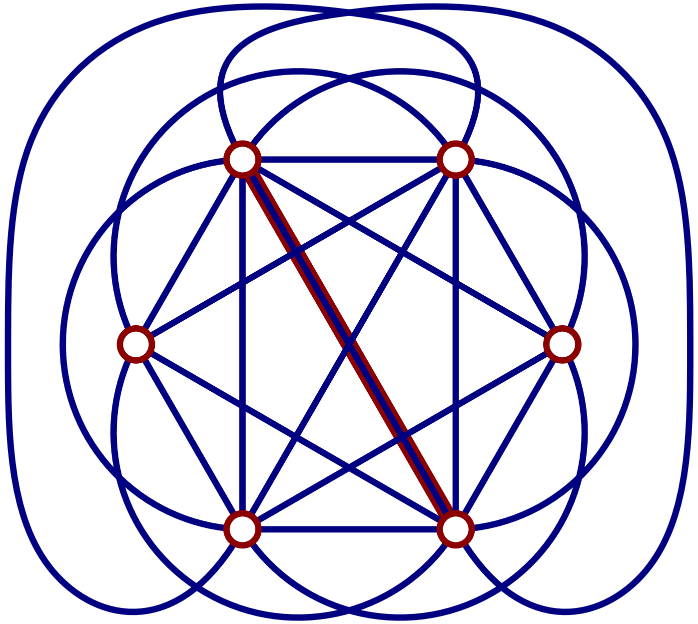

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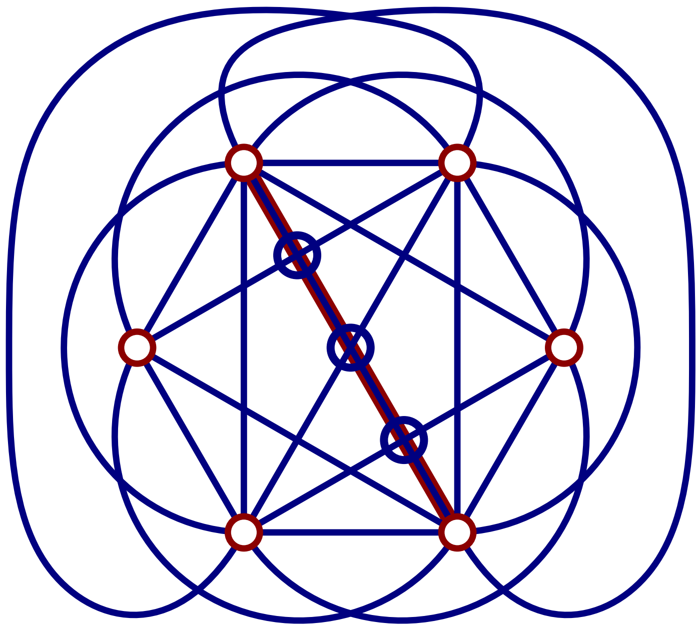

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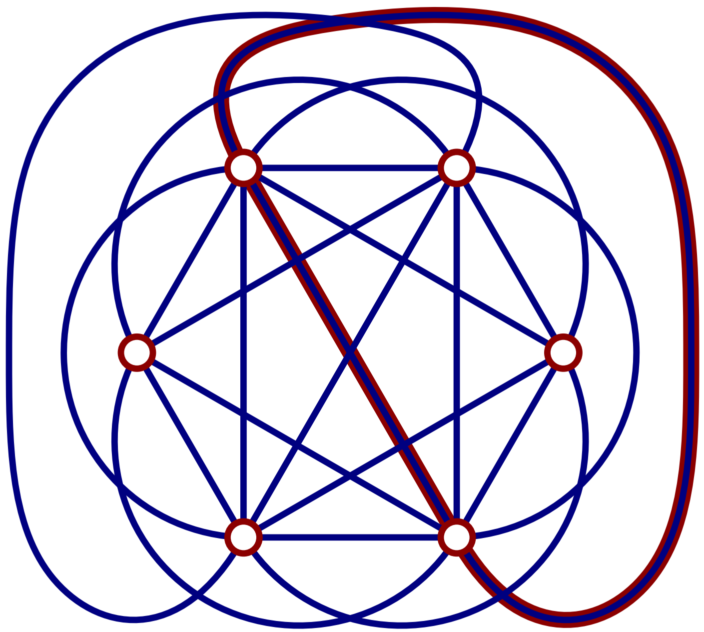

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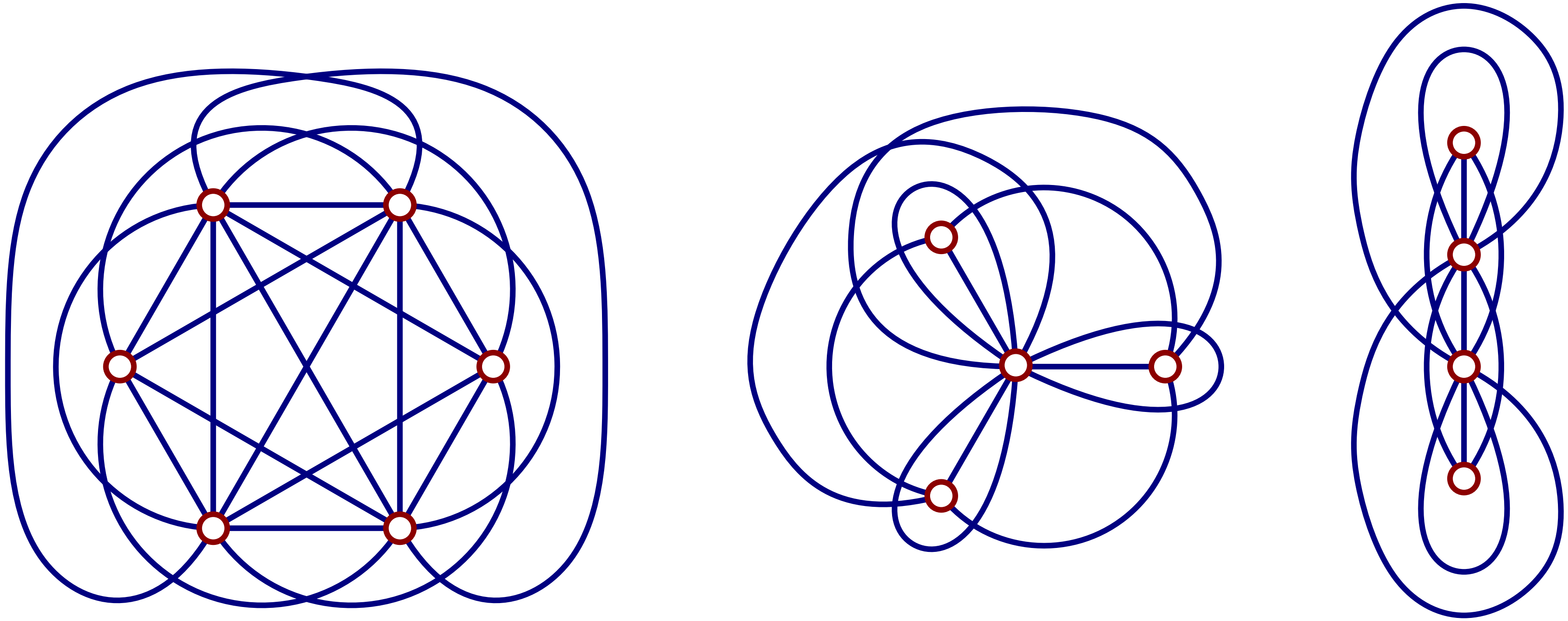

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**Theorem:** Non simple 3-planar graphs on $n$ vertices have at most $5.5n - 11$ edges.
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**Proof:**

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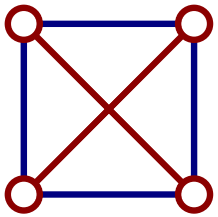
- For simple graphs the same bound holds
- The bound is not known to be tight

# Lower Bound

1-planar

$K_4$



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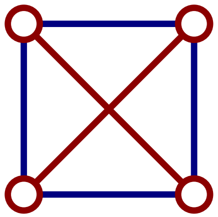
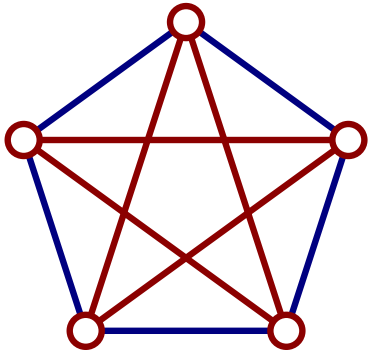


2-planar

$K_5$



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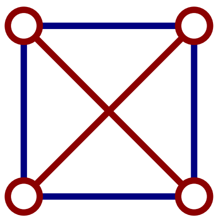

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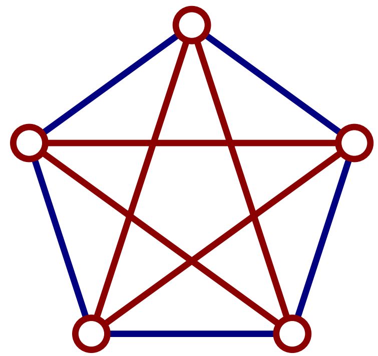

3-planar

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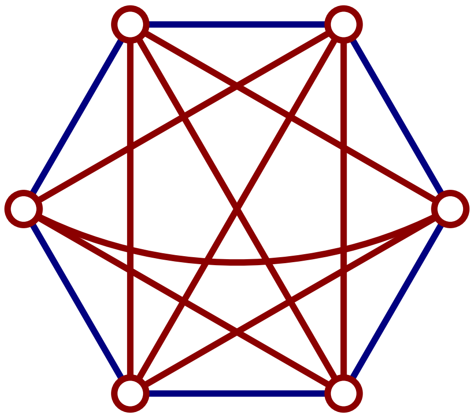

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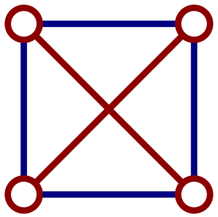

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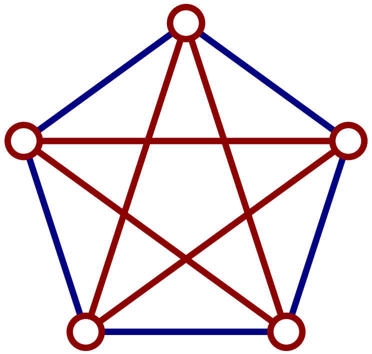

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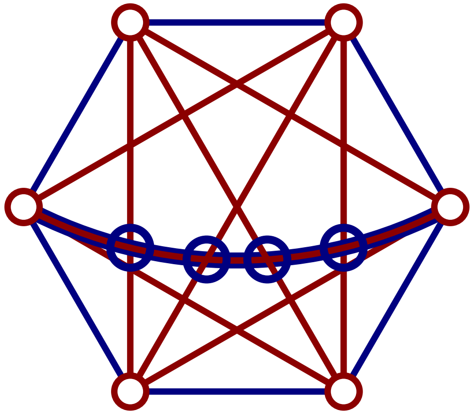

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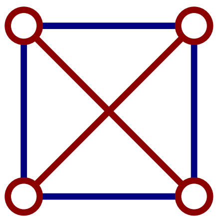

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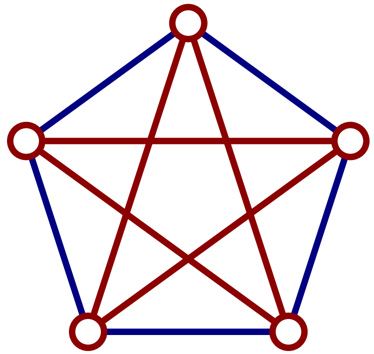

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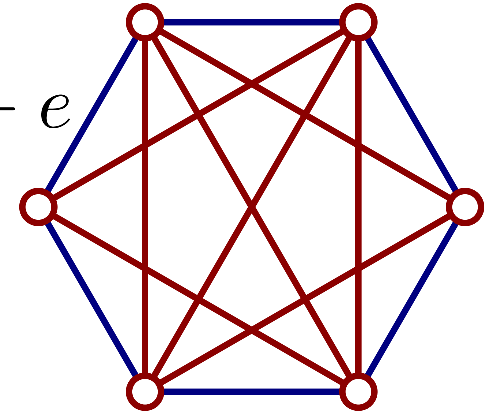
$K_6 - e$



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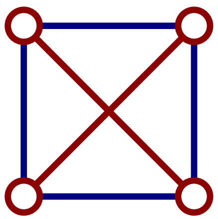

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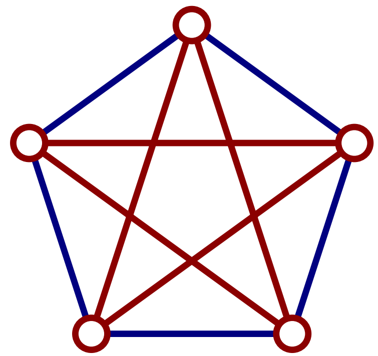

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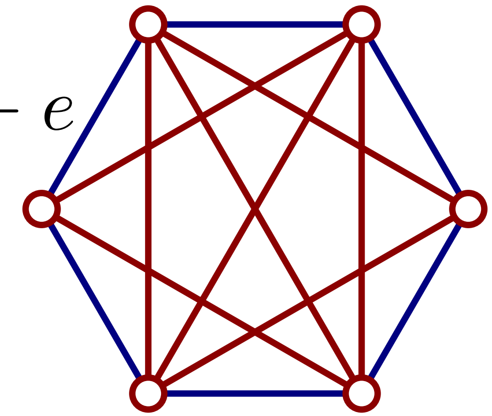

Planar graph with faces of length 6

# Lower Bound

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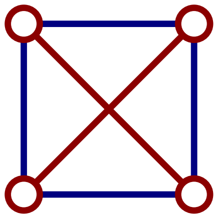

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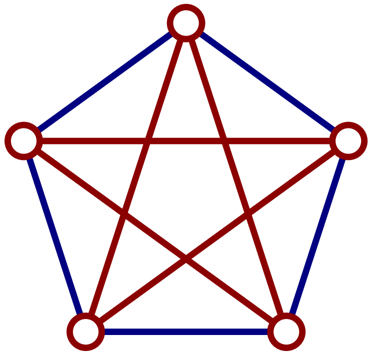

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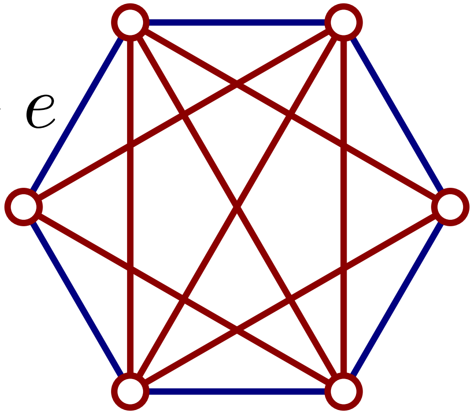

Planar graph with faces of length 6

- Euler's formula: $n - 2 = e_p - f$

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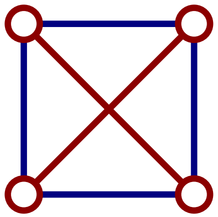

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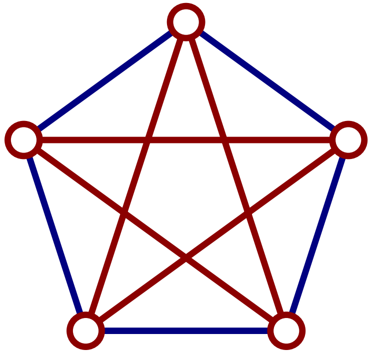

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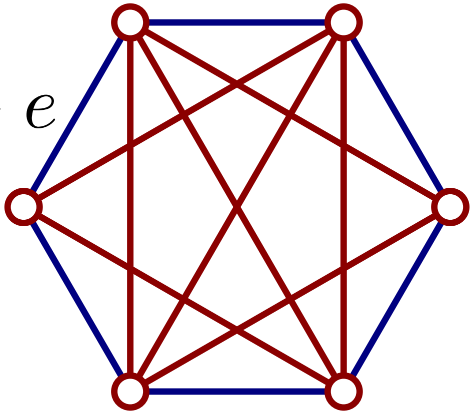

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- Euler's formula: $n - 2 = e_p - f$
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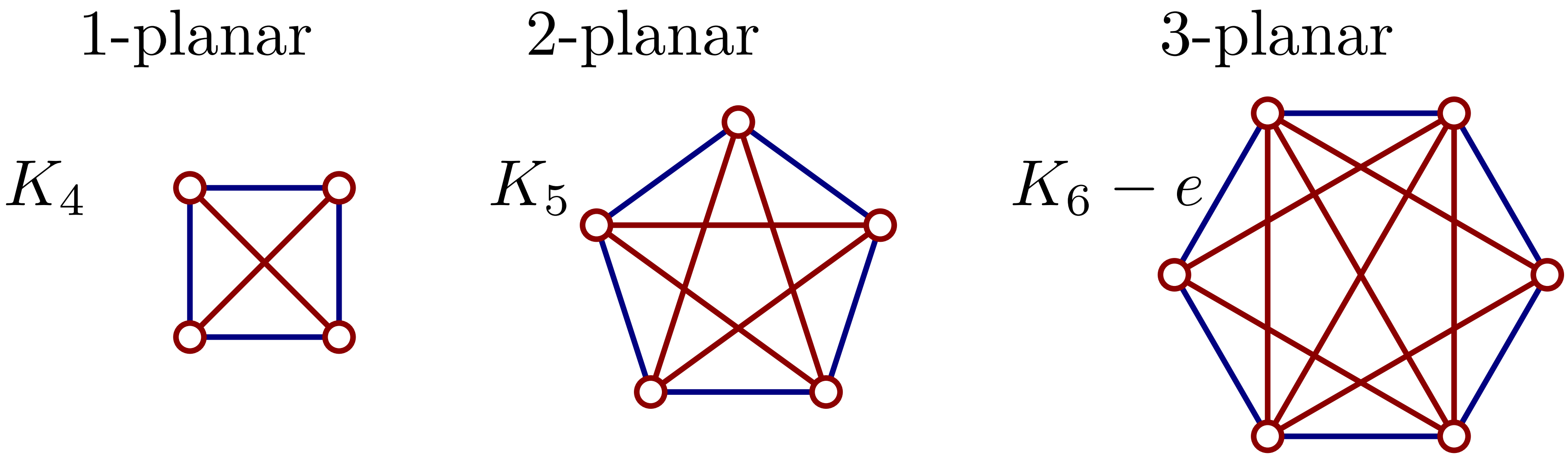


Planar graph with faces of length 6

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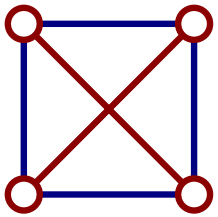

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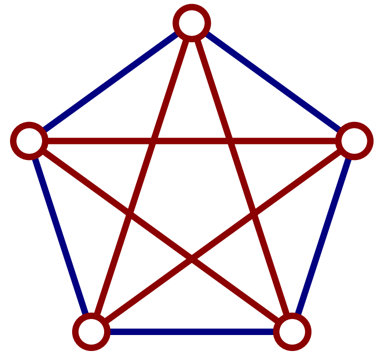

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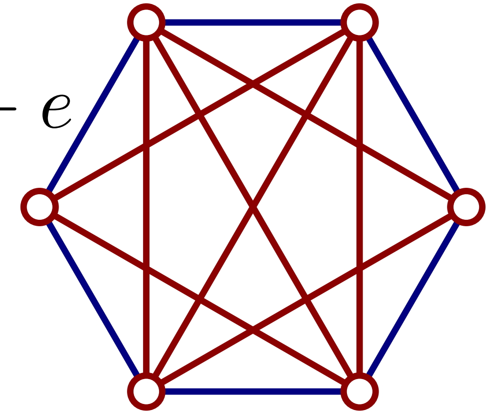

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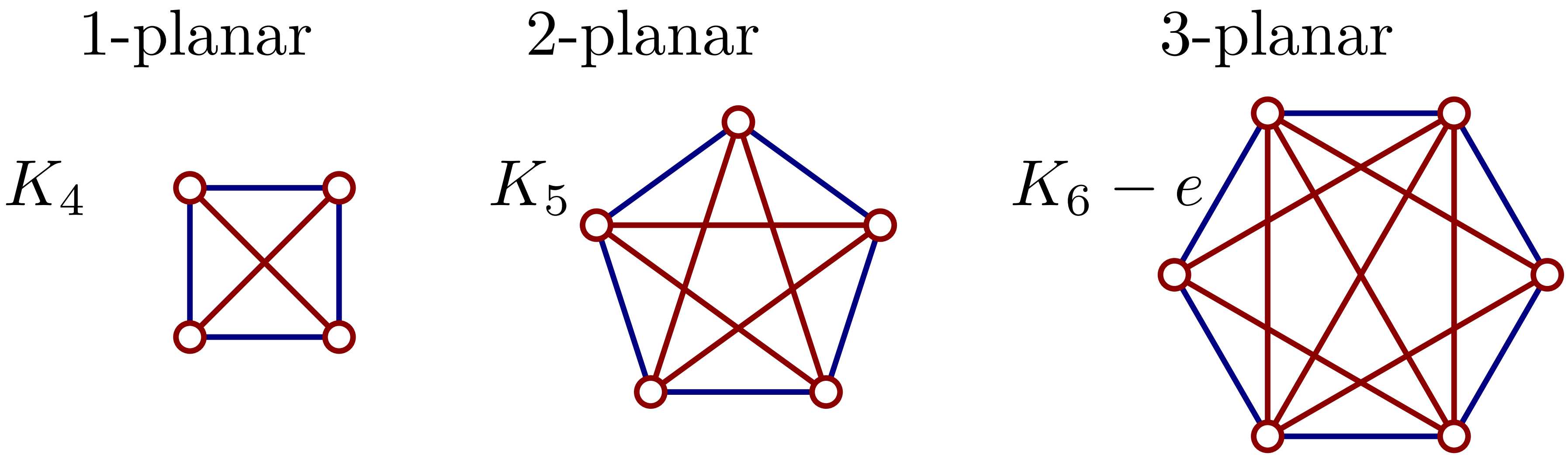


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- Total edges: $e = e_p + e_c = 11f$

$$\Rightarrow e = \frac{11}{2}(n-2) = 5.5n - 11$$

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Construction of Pach et al.

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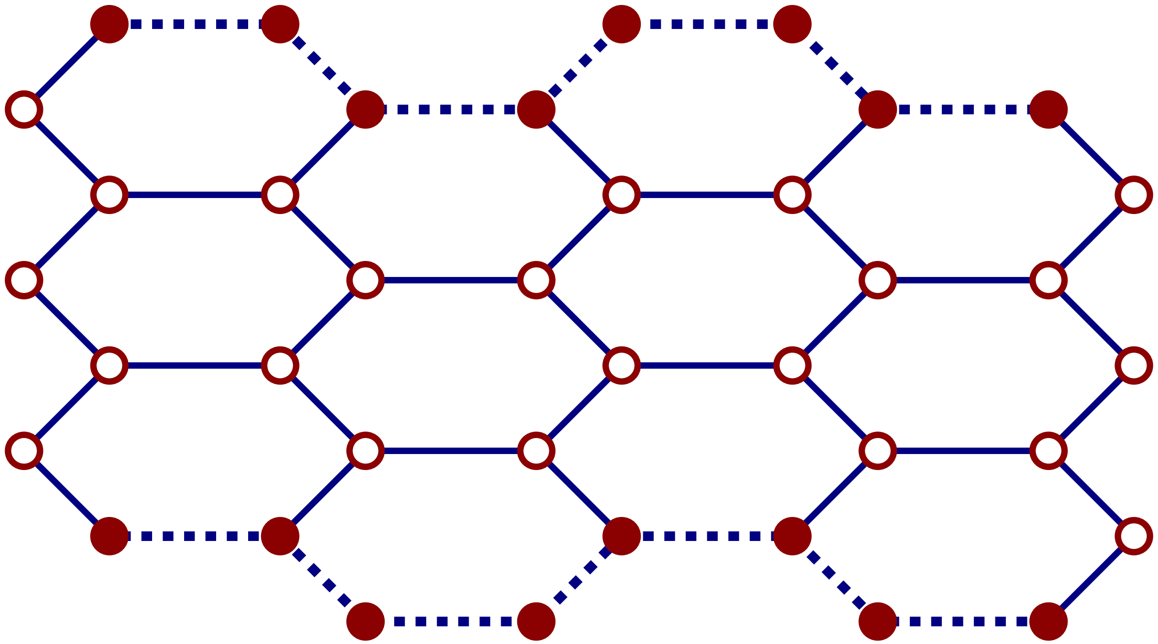

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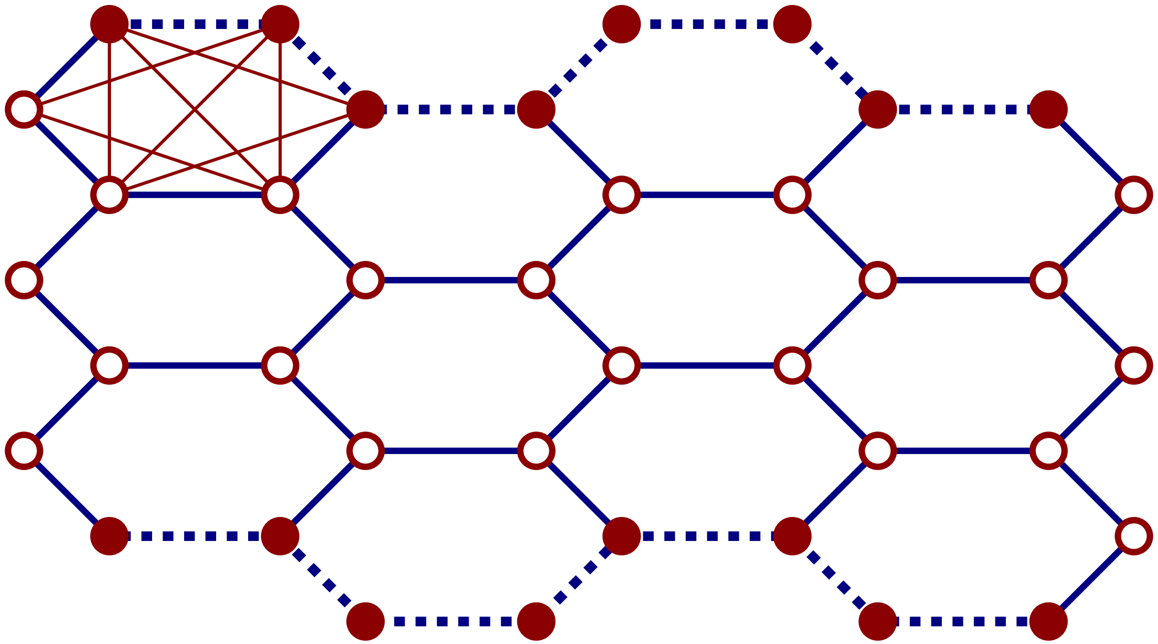

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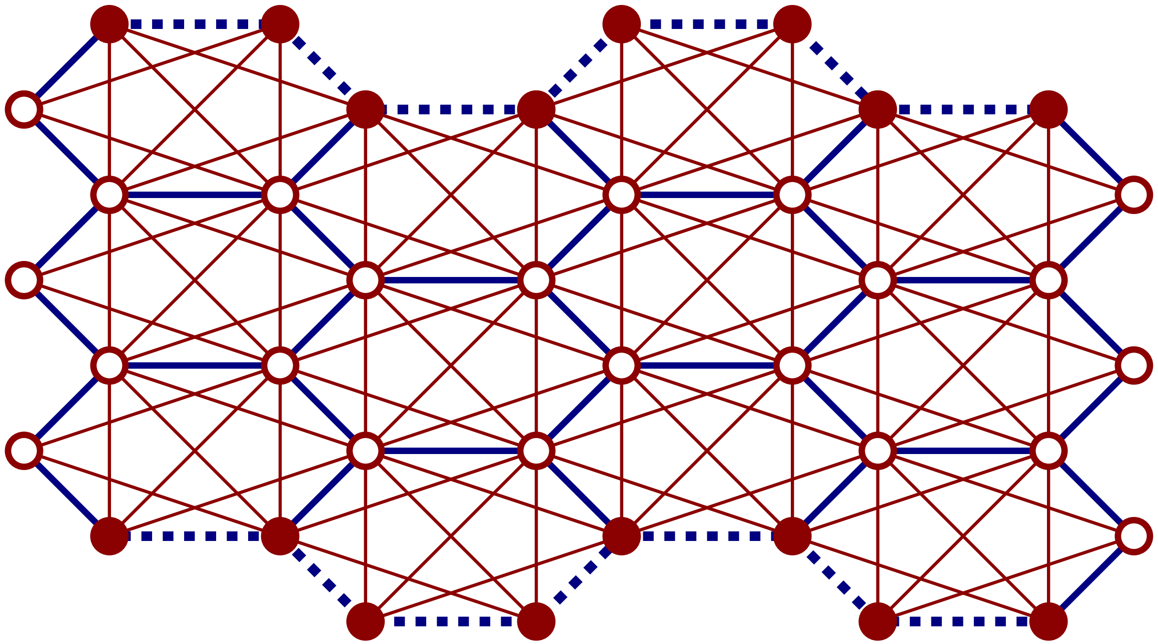
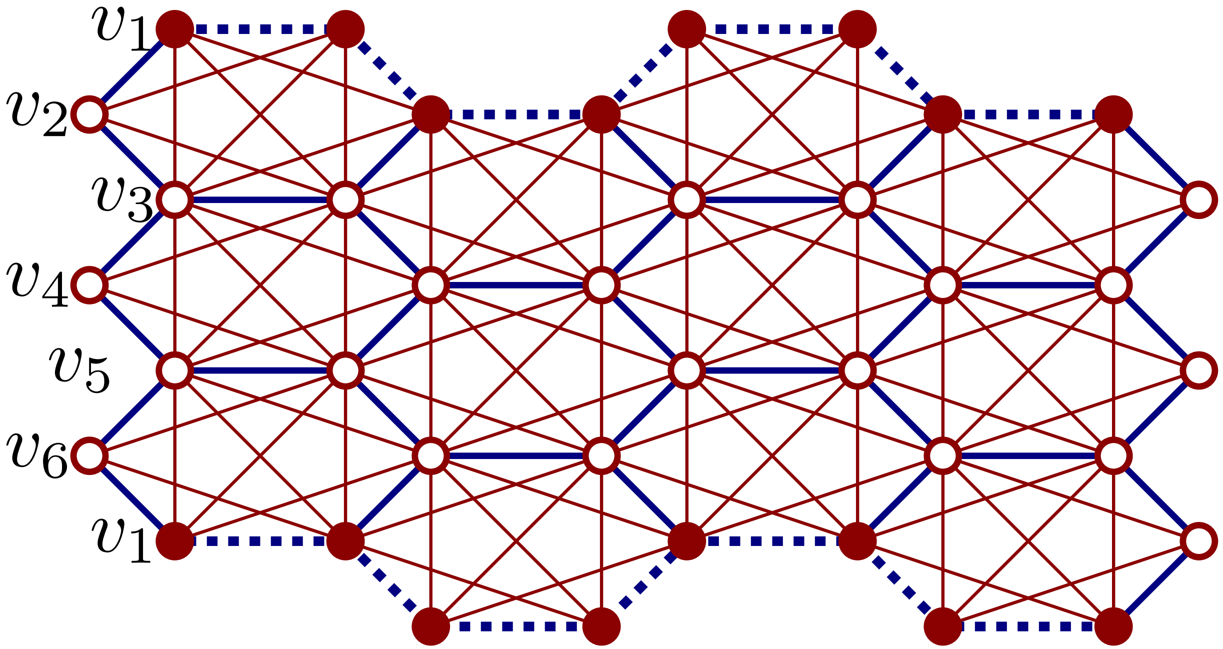


# Lower Bound

Construction of Pach et al.



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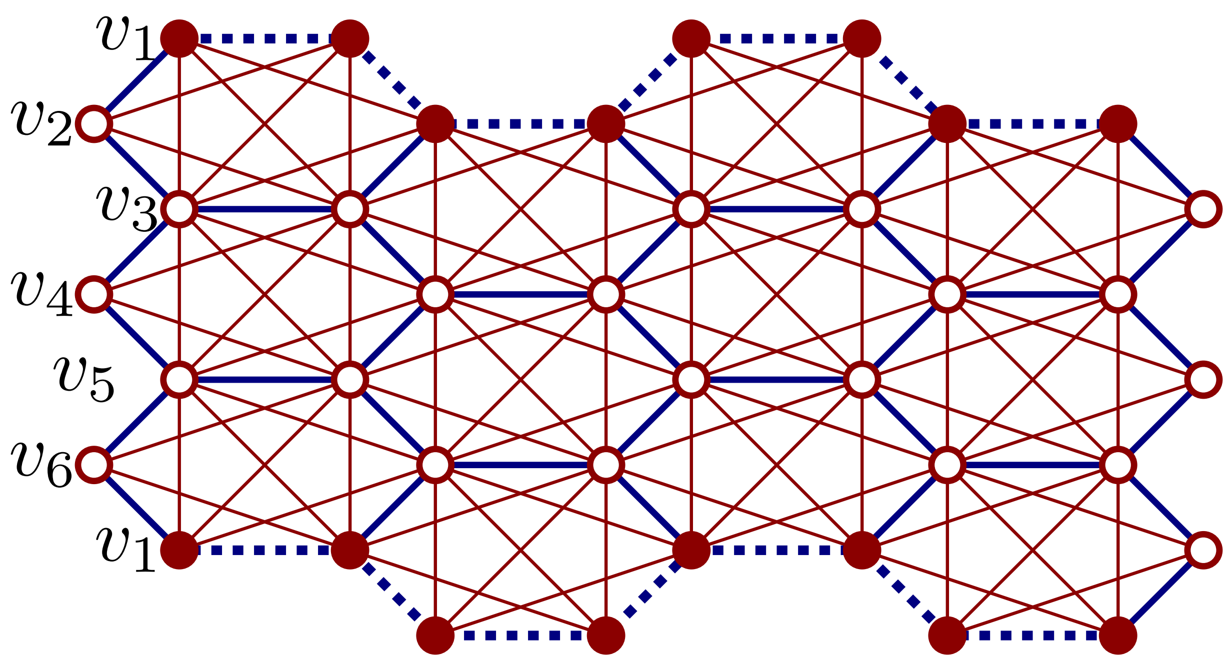


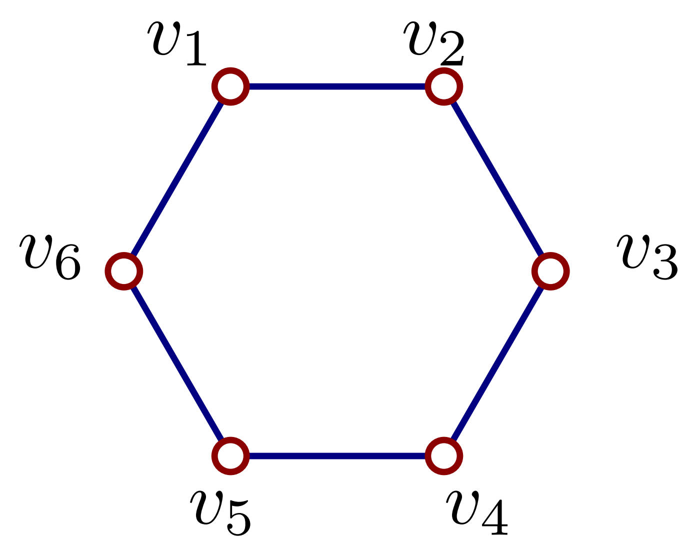

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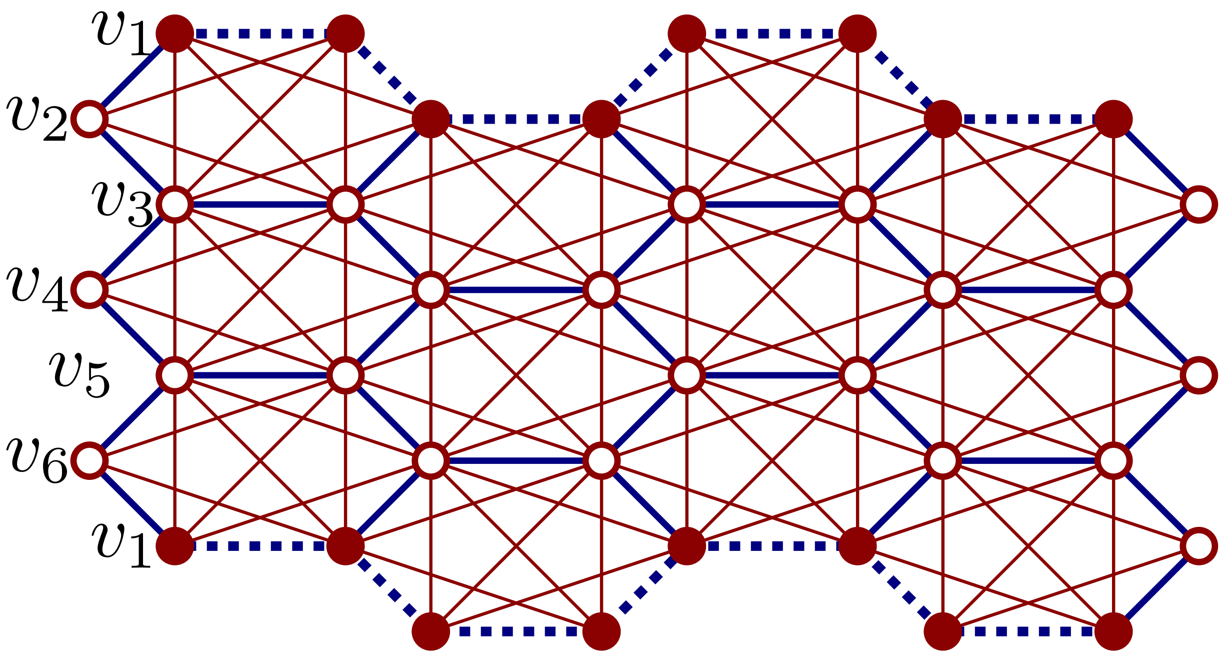


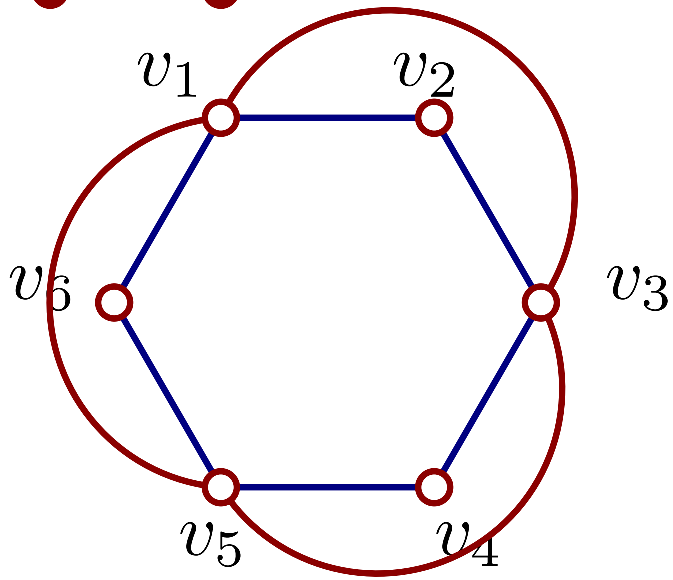

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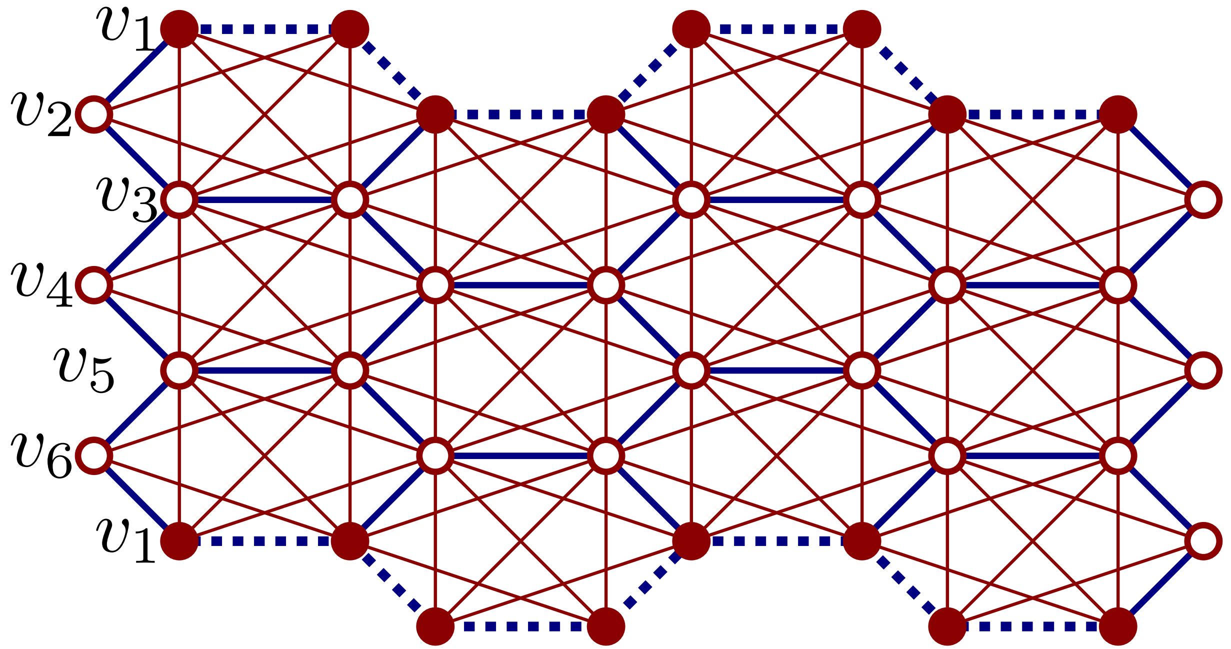


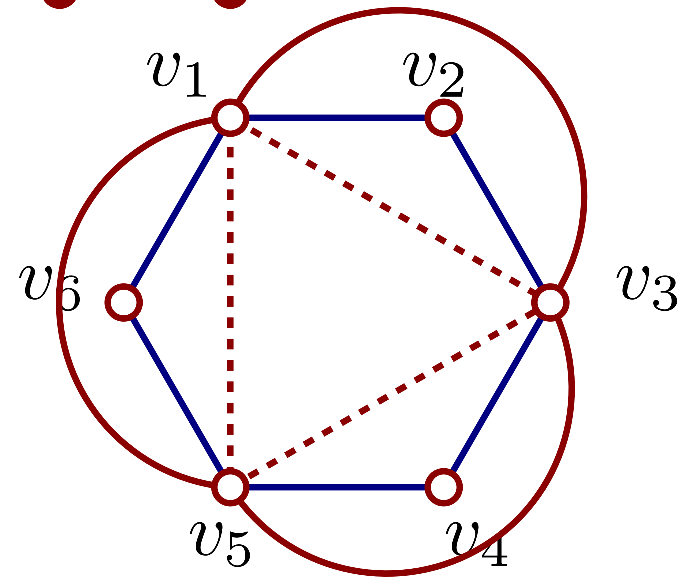

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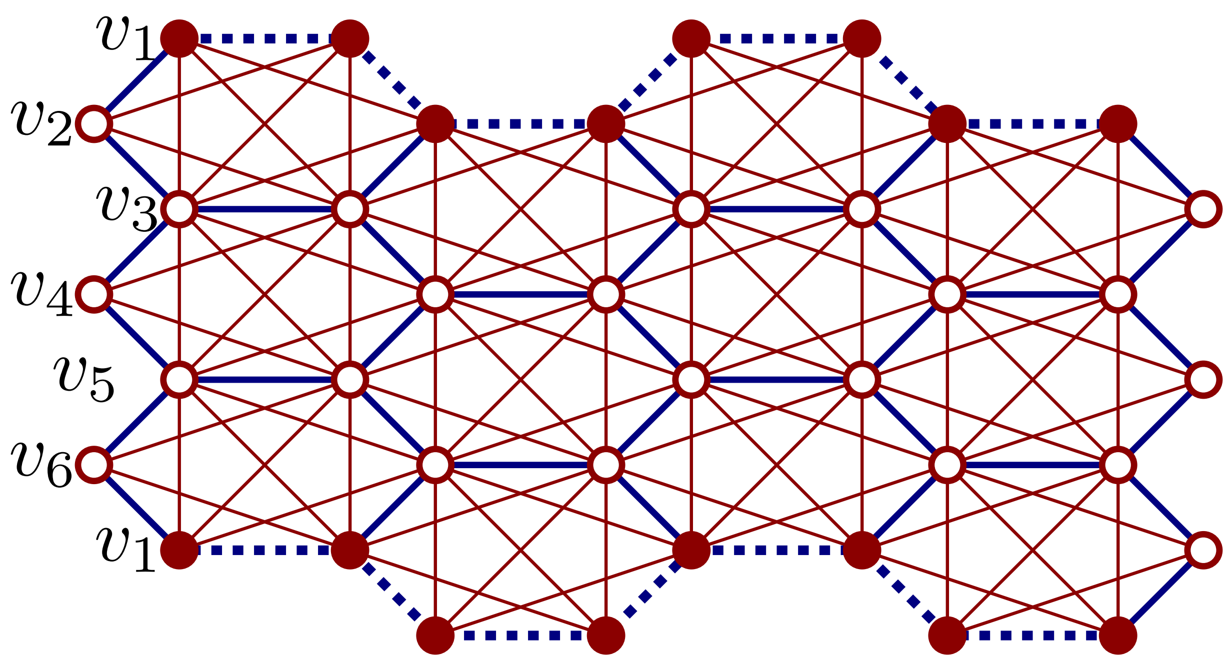
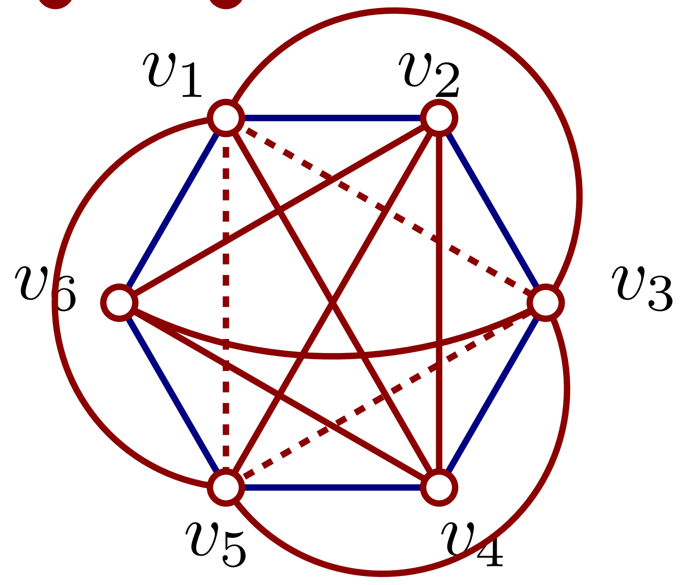



6 edges



# Lower Bound

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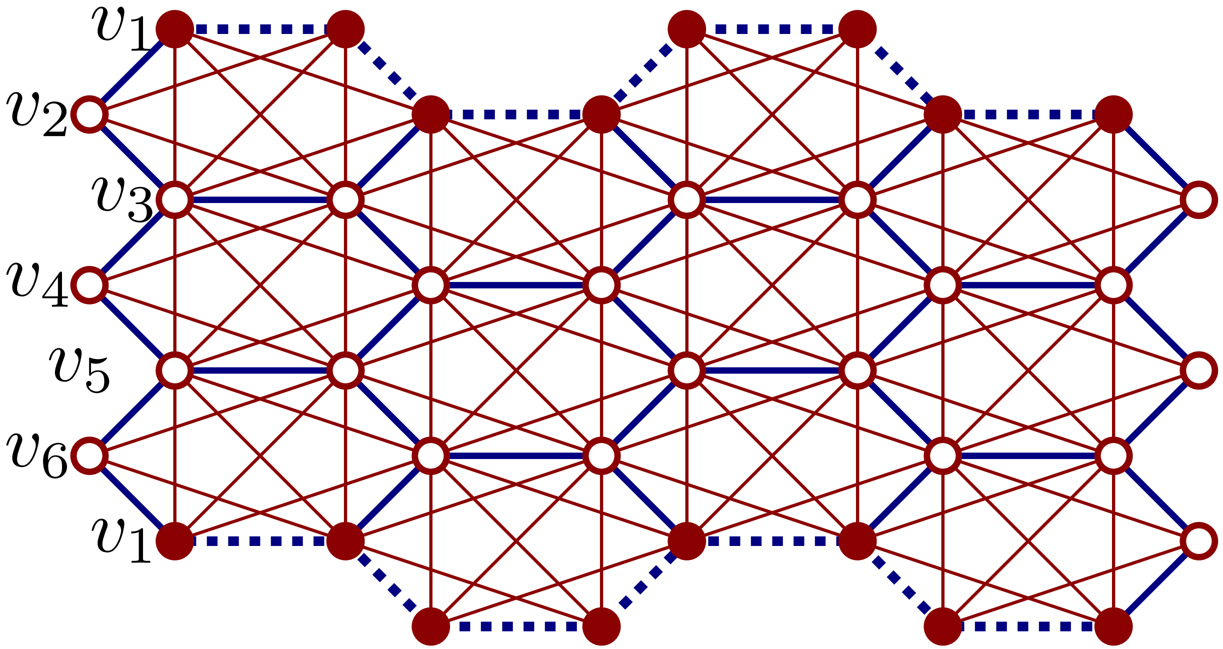
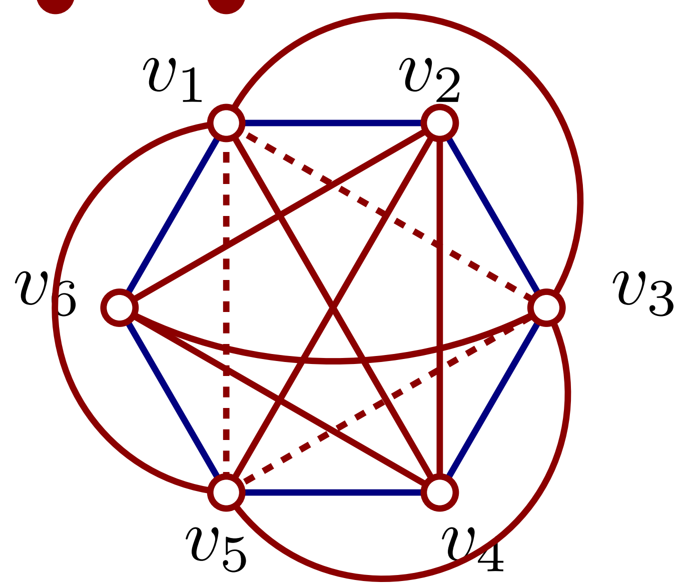



6 edges

5.5$n$ − 15 edges



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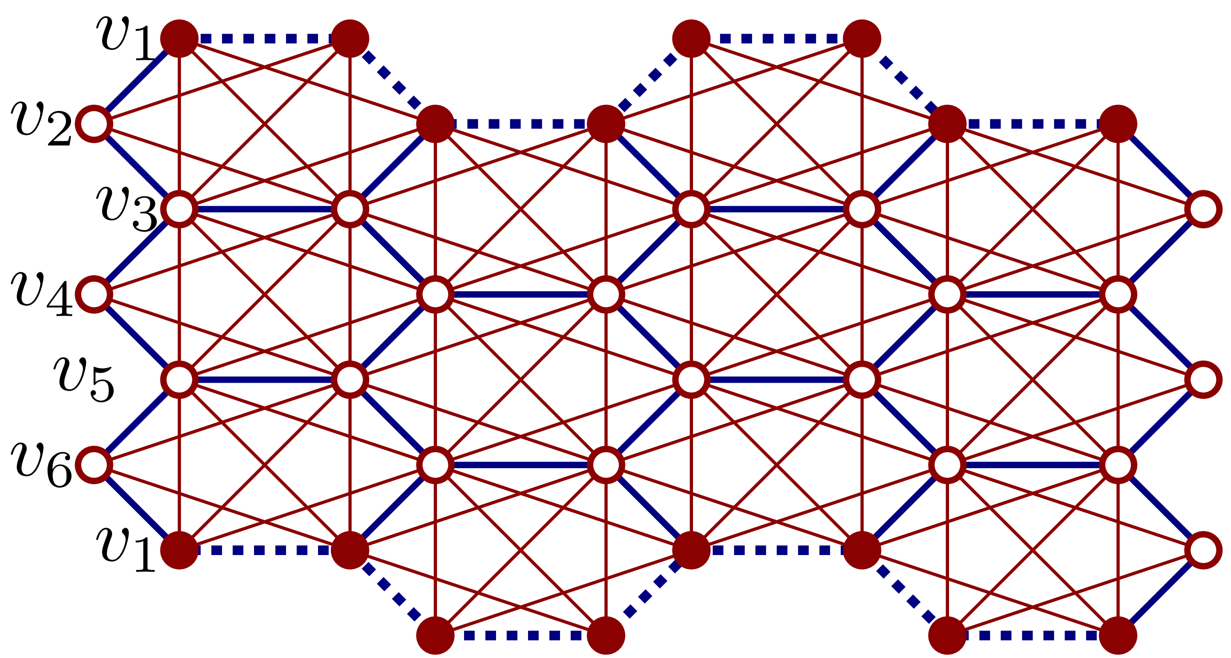
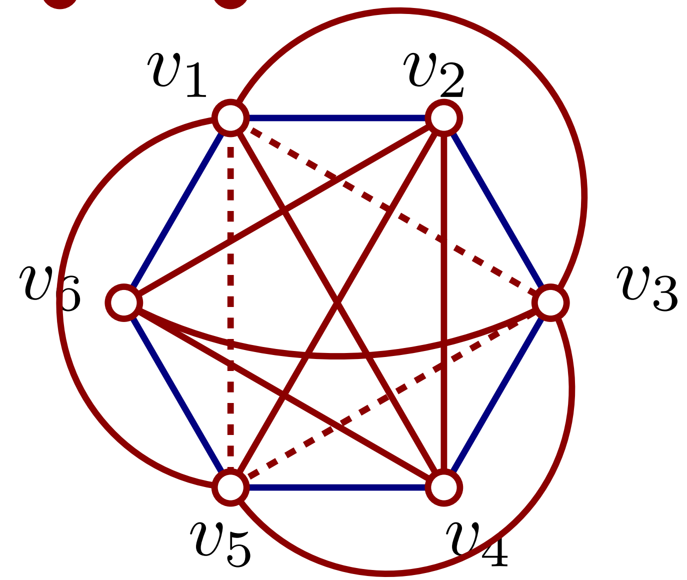



6 edges

$5.5n - 15$ edges

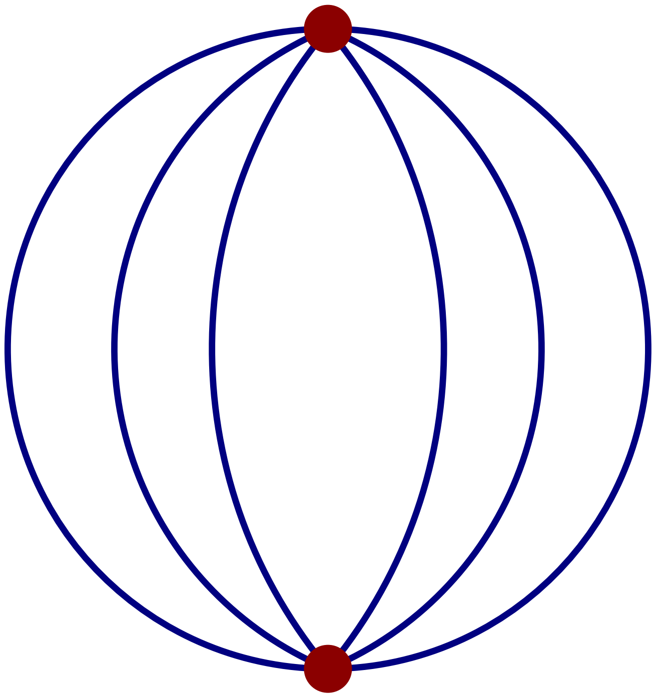
$(5.5n - 11)$

# Lower Bound

Other constructions:

# Lower Bound

Other constructions:



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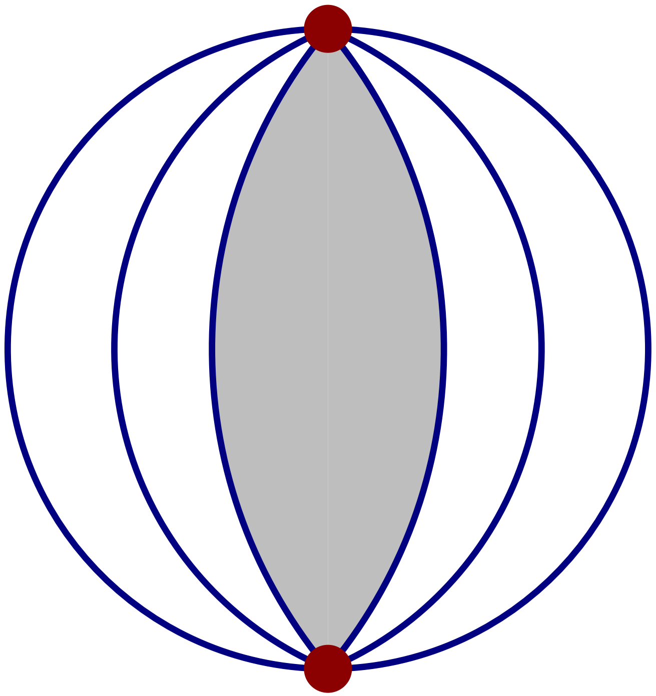

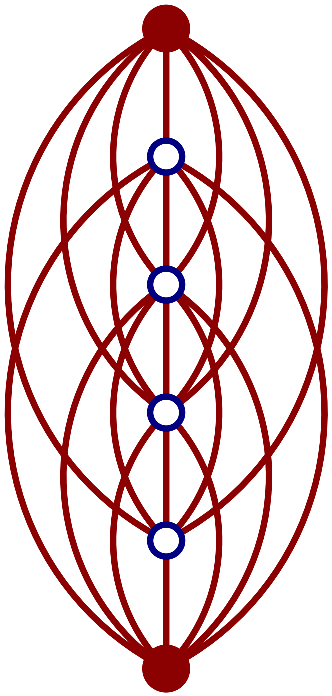

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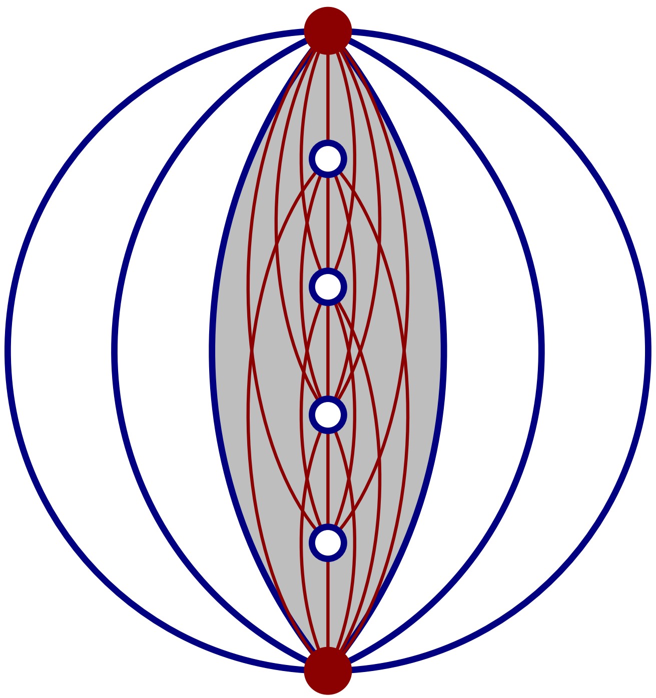

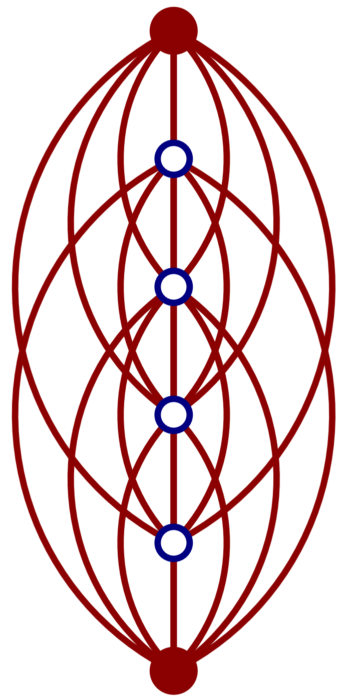

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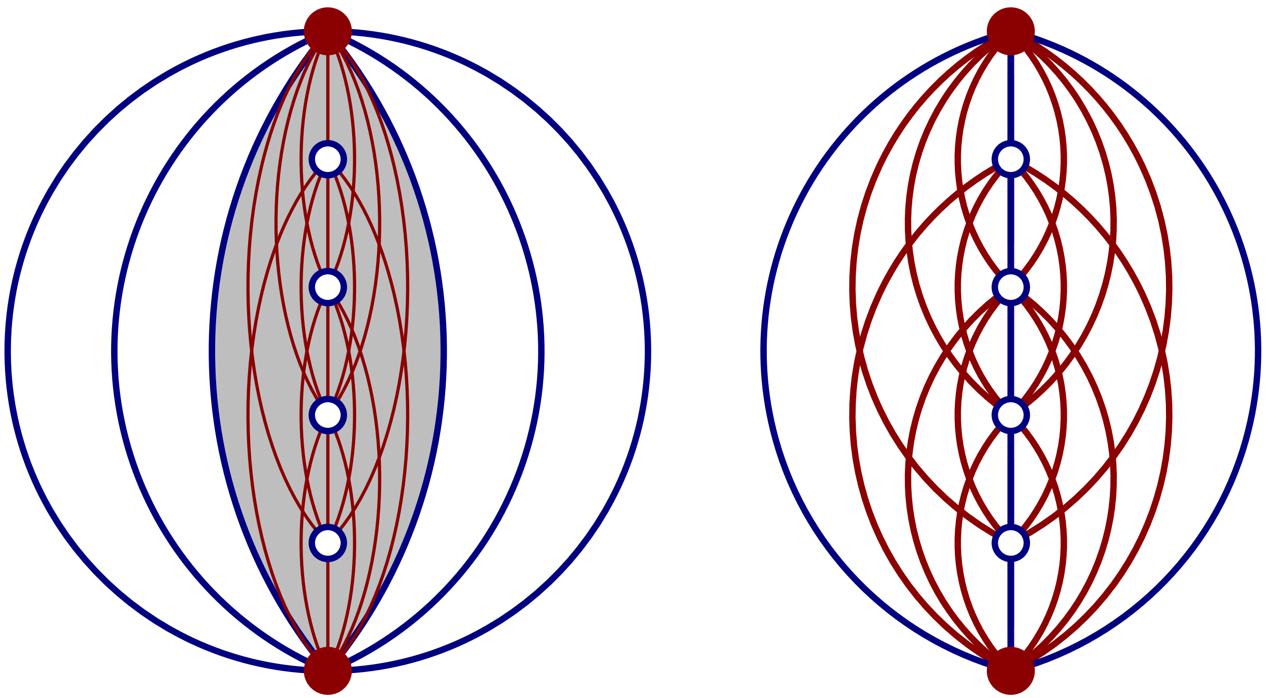


two 6-gons
8 interior edges

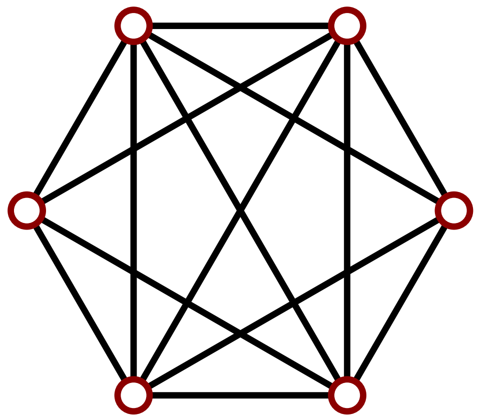
# Upper Bound

## General Idea:

# Upper Bound

## General Idea:

- Plane subgraph $G_p$
  - spanning, connected
  - maximum number of edges $e_p$



# Upper Bound

## General Idea:

- Plane subgraph $G_p$
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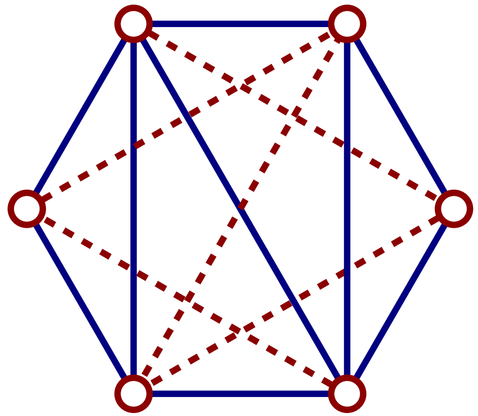
  - maximum number of edges $e_p$

*planar edges:* $e_p \leq 3n - 6$



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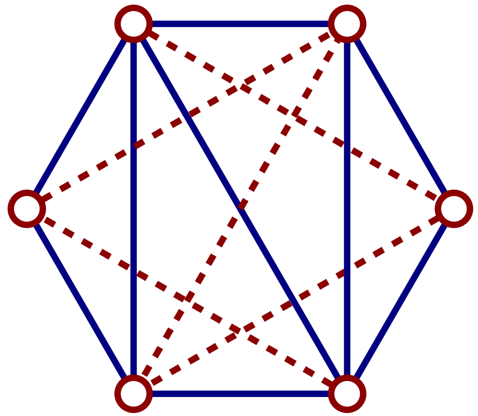

- Bound the remaining edges $e_c$

    *crossing edges:* $e_c$

  - crossing edges cross with planar edges

# Upper Bound

- Maximal plane subgraph $G_p$
  - triangular faces
  - $e_p = 3n - 6$

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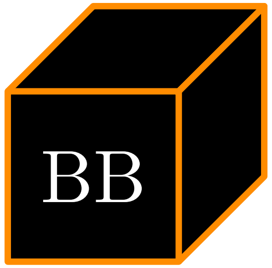

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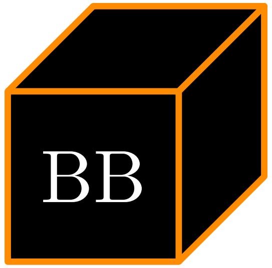


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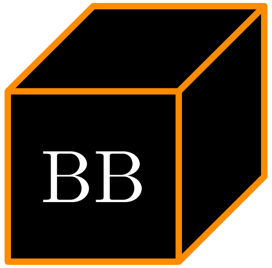
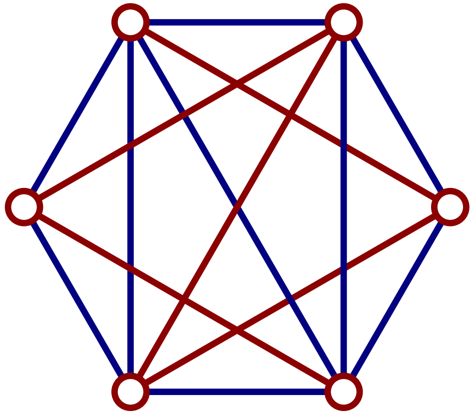



- crossing edges



- a crossing edge consists of *sticks* and *middle parts*

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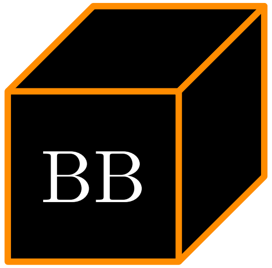
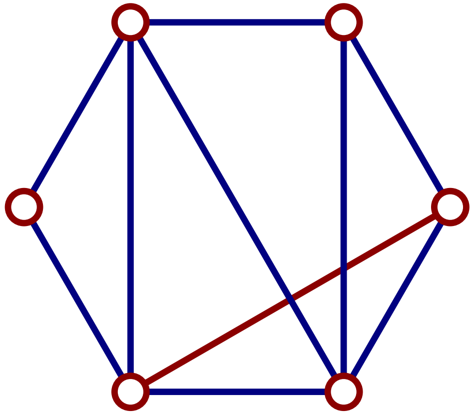



- crossing edges



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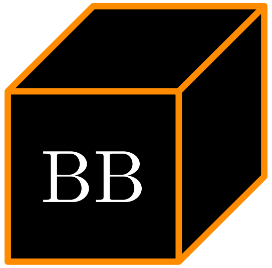
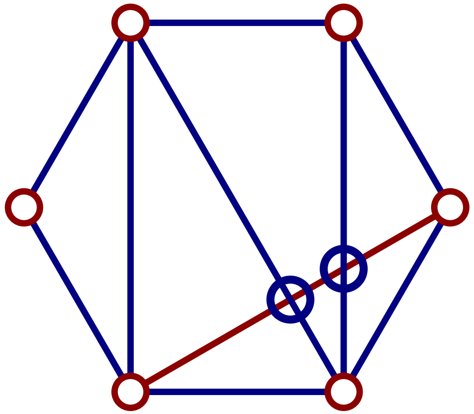



- crossing edges



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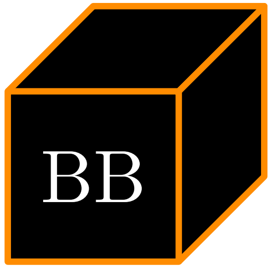


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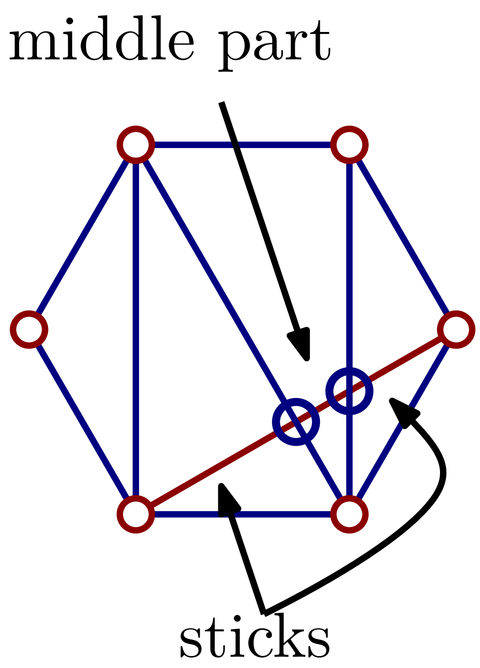


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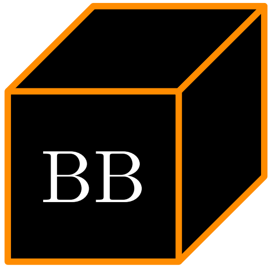
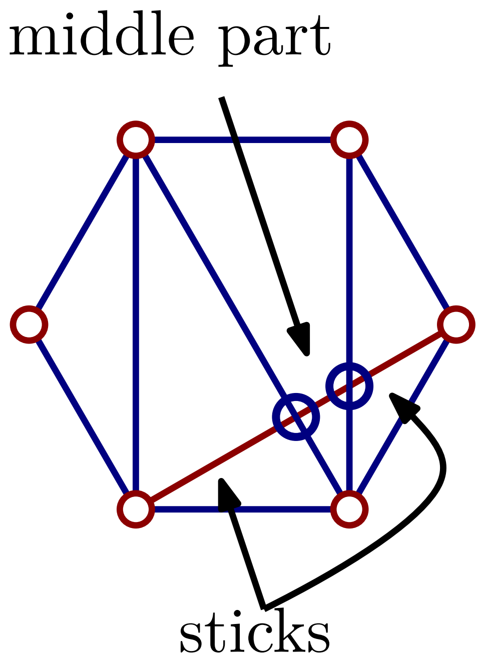



- crossing edges



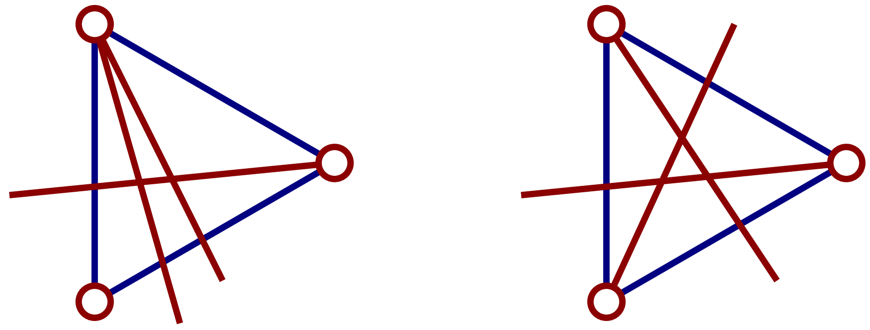

- a crossing edge consists of *sticks* and *middle parts*
- sticks lie inside faces
- count sticks:
  $\Rightarrow 2e_c$=#STICKS

# Upper Bound

A triangular face has at most 3 sticks

# Upper Bound

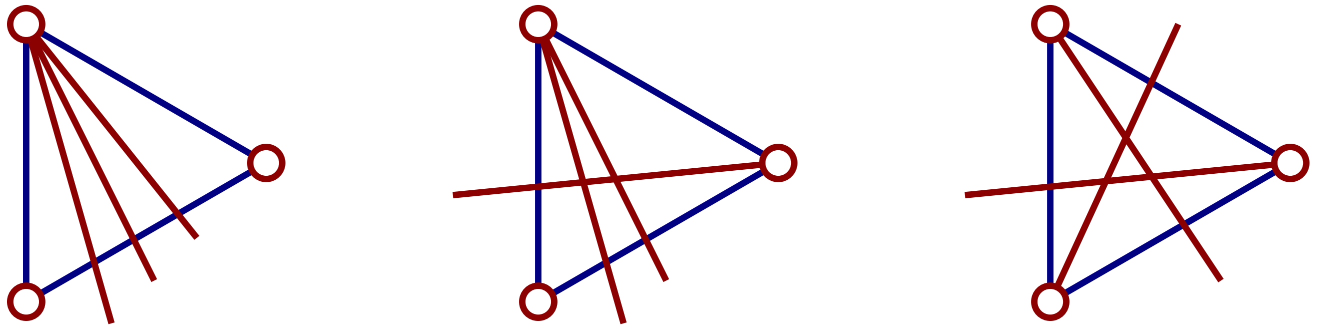
A triangular face has at most 3 sticks



$(3, 0, 0)$ $(2, 1, 0)$ $(1, 1, 1)$

# Upper Bound

A triangular face has at most 3 sticks



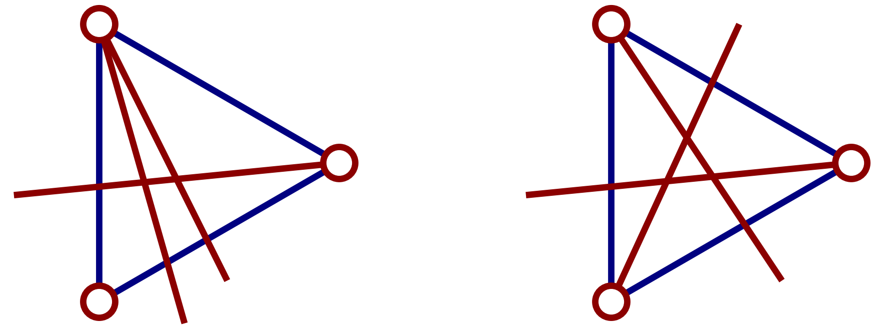
$(3,0,0)$ $(2,1,0)$ $(1,1,1)$

- If *all faces* have 3 sticks

$$e_c = 3f/2 = 3(n-2)$$
$$e = e_p + e_c = 6(n-2)$$
$$(e_p = 3n-6)$$

# Upper Bound

A triangular face has at most 3 sticks



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- If *all faces* have 3 sticks

  $e_c = 3f/2 = 3(n-2)$
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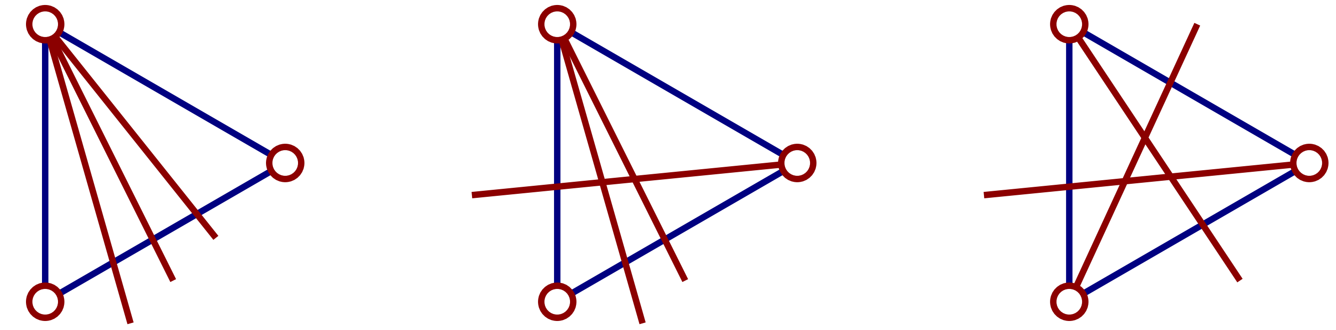
- **Total:** $6n - 12$

# Upper Bound

A triangular face has at most 3 sticks



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- If *all faces* have 3 sticks

  $e_c = 3f/2 = 3(n-2)$
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  $(e_p = 3n-6)$

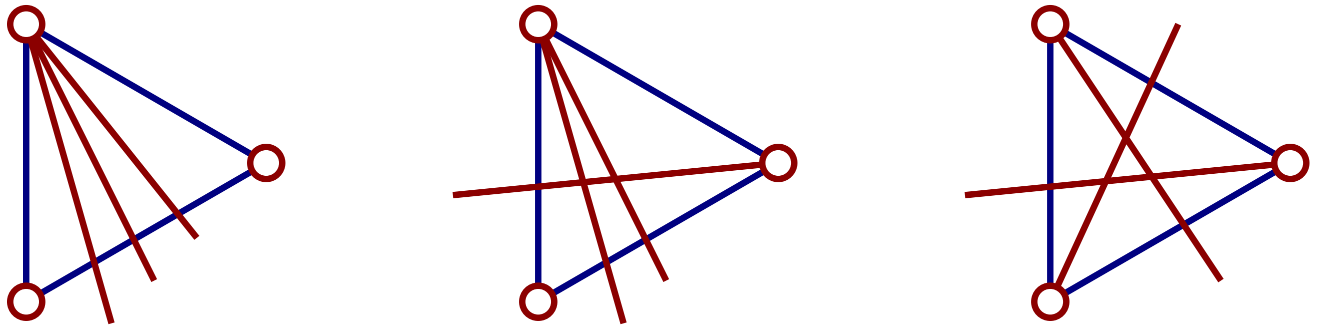
- If *half of faces* have 3 sticks

  $e_c = 3f/4 + 2f/4 = 5(n-2)/2$
  $e = e_p + e_c = 11(n-2)/2$

- **Total:** $6n - 12$

# Upper Bound

A triangular face has at most 3 sticks



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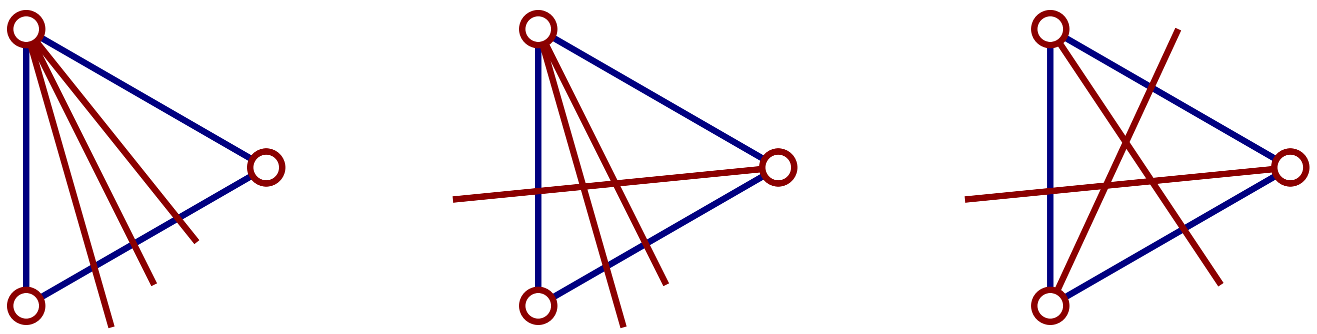
- **Total:** $5.5n - 11$

# Upper Bound

A triangular face has at most 3 sticks



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- **Prove that at most half faces have 3 sticks**

# Upper Bound

## At most half faces have 3 sticks

- Associate (uniquely) a face with 3 sticks with a face with at most 2 sticks

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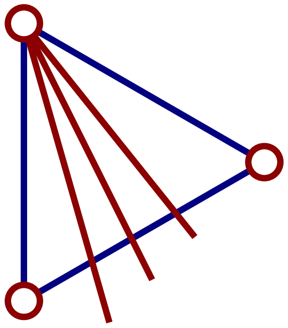
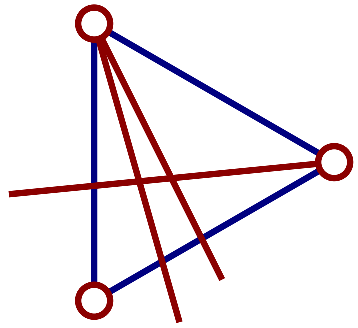
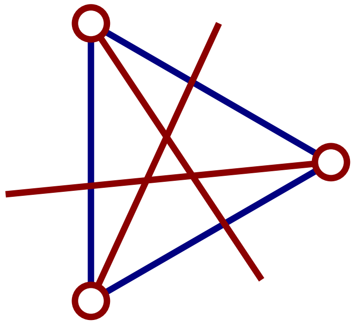


$(3, 0, 0)$



$(2, 1, 0)$

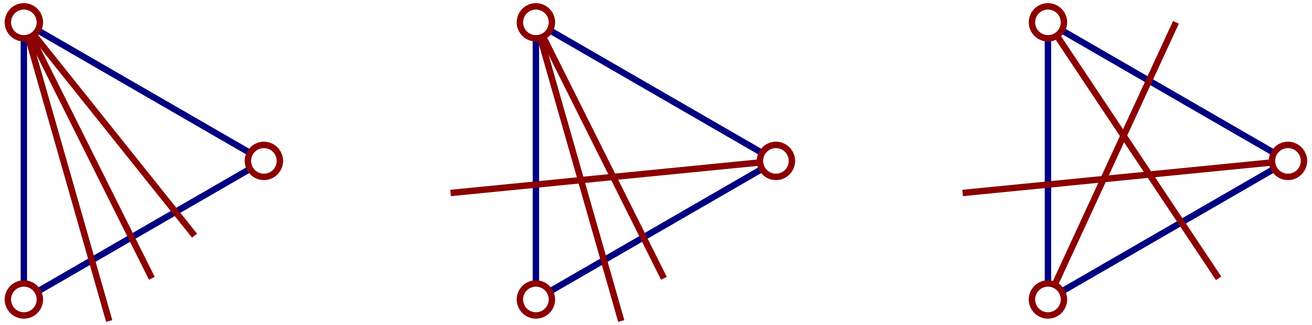


$(1, 1, 1)$

# Upper Bound

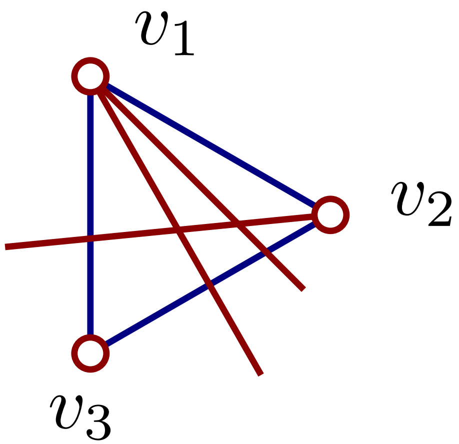
## At most half faces have 3 sticks

- Associate (uniquely) a face with 3 sticks with a face with at most 2 sticks



$(3,0,0)$ $(2,1,0)$ $(1,1,1)$

$(2,1,0)$



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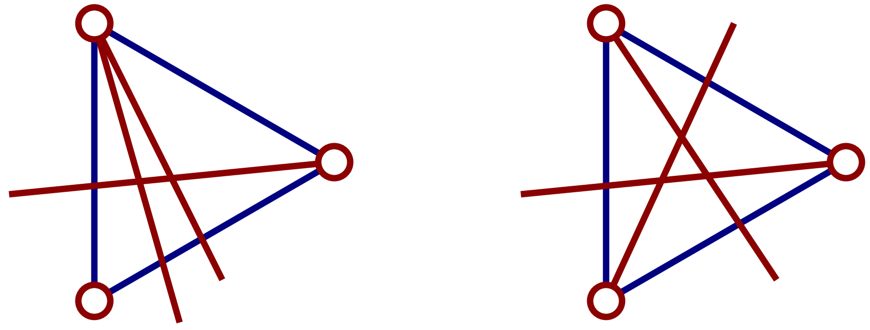

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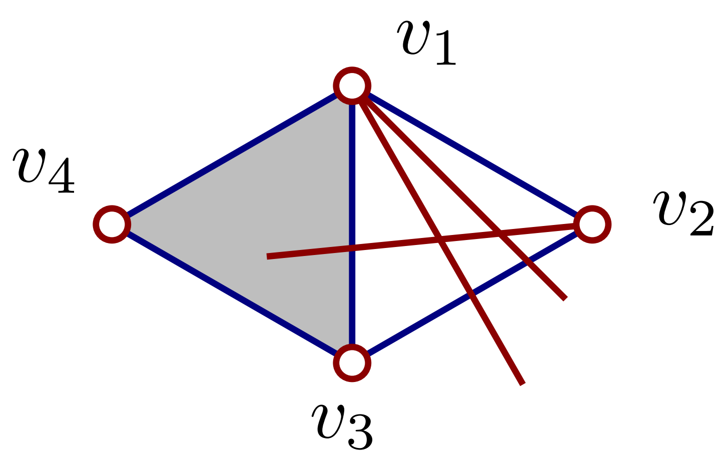

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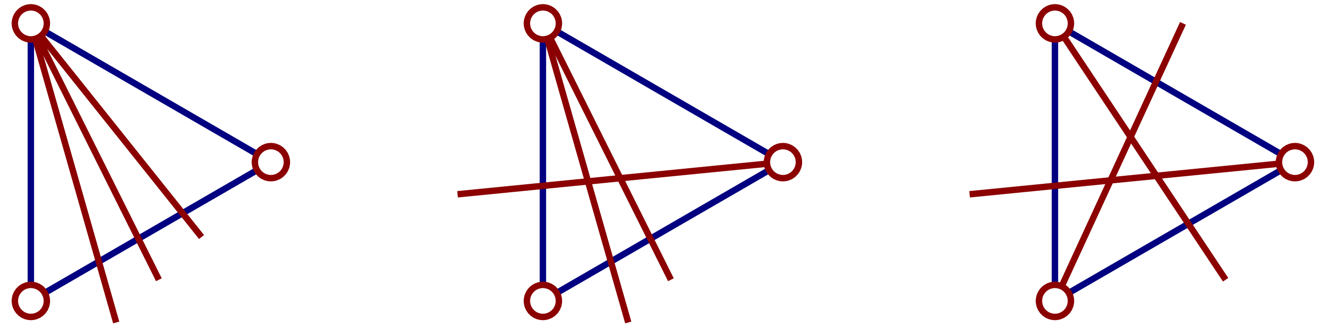

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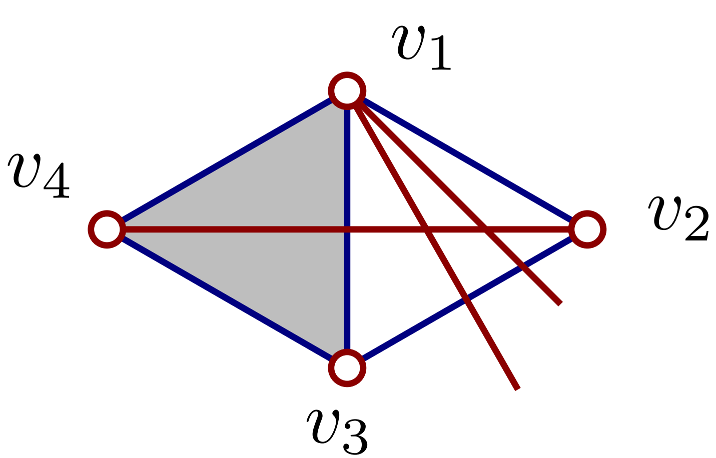

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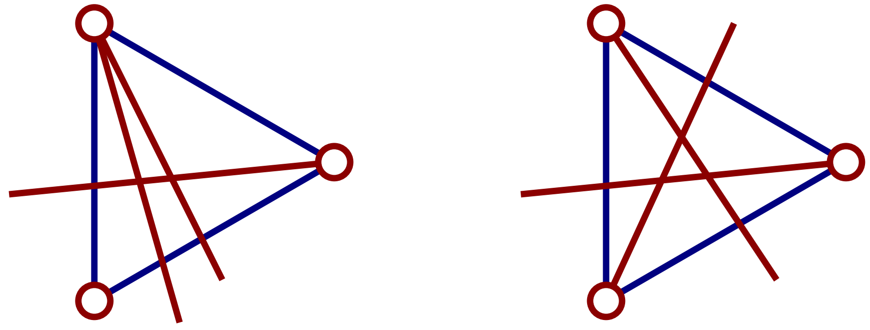

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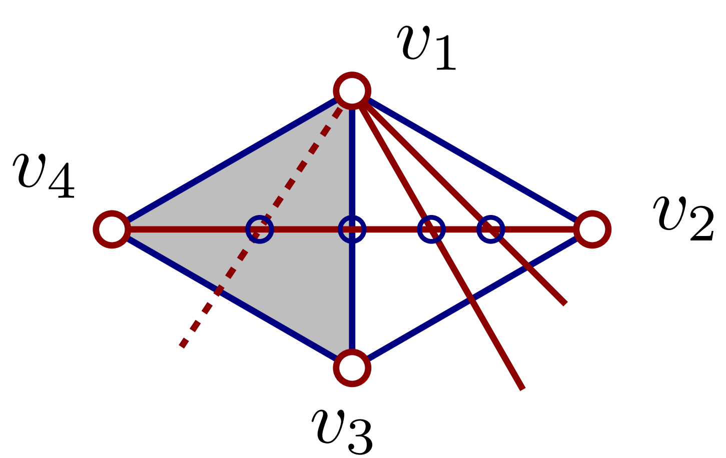

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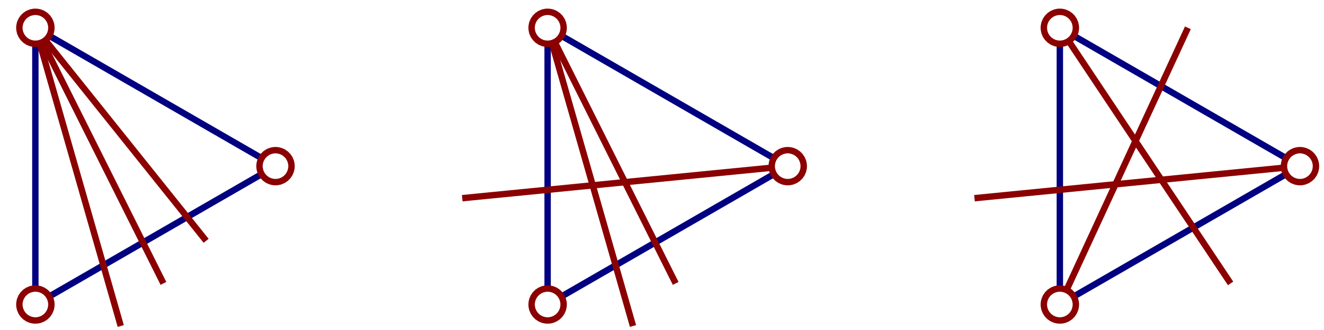

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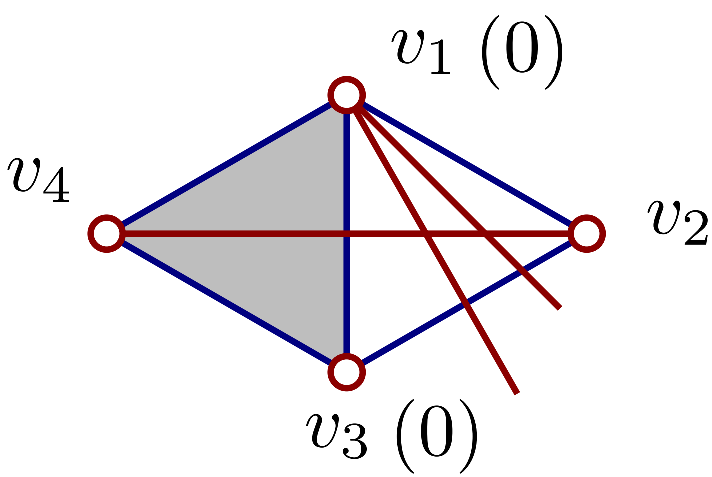

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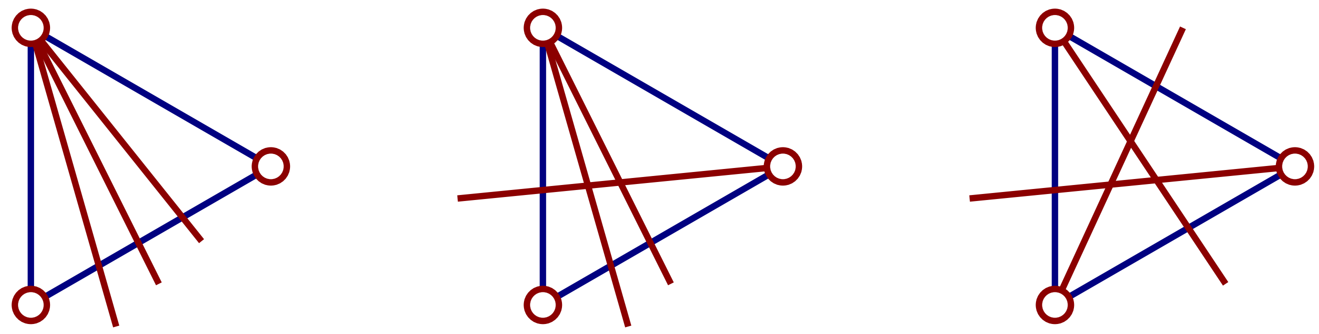

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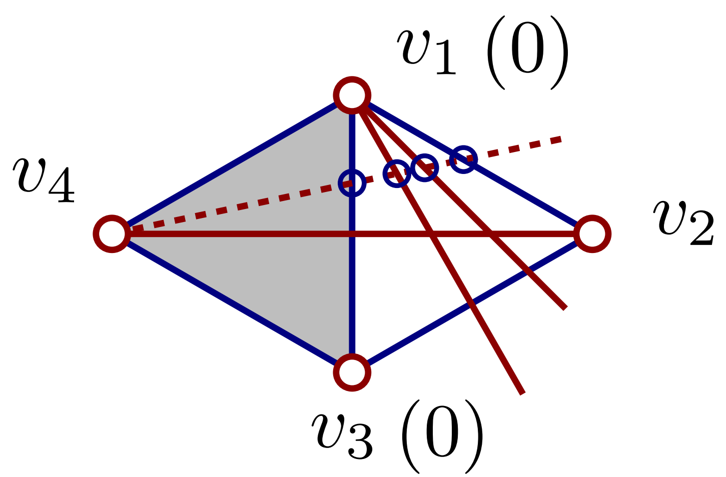

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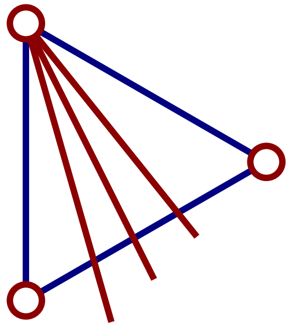
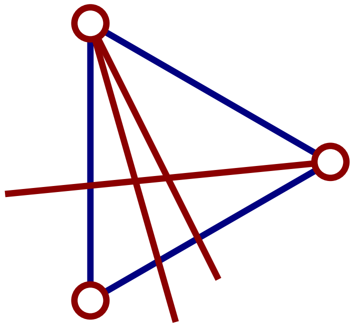
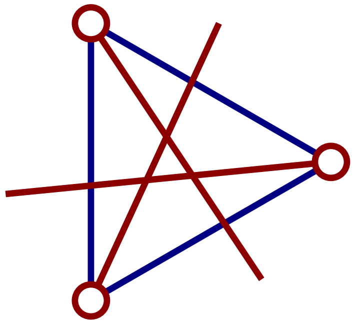


$(3,0,0)$

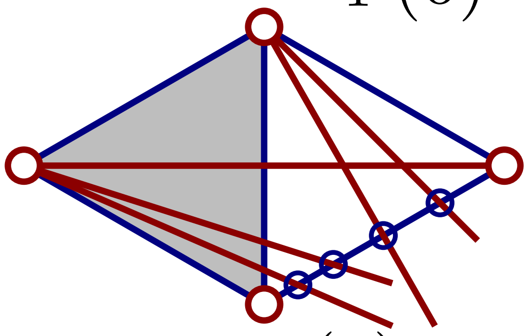


$(2,1,0)$



$(1,1,1)$

$(2,1,0)$



# Upper Bound

## At most half faces have 3 sticks

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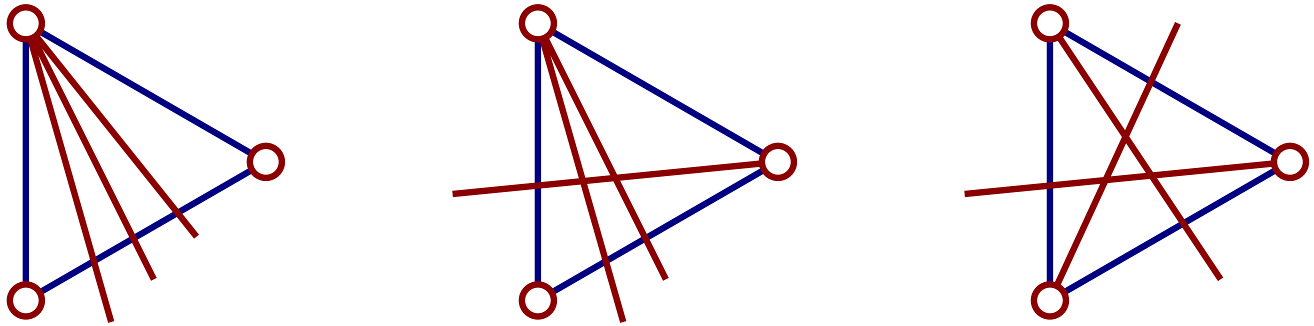
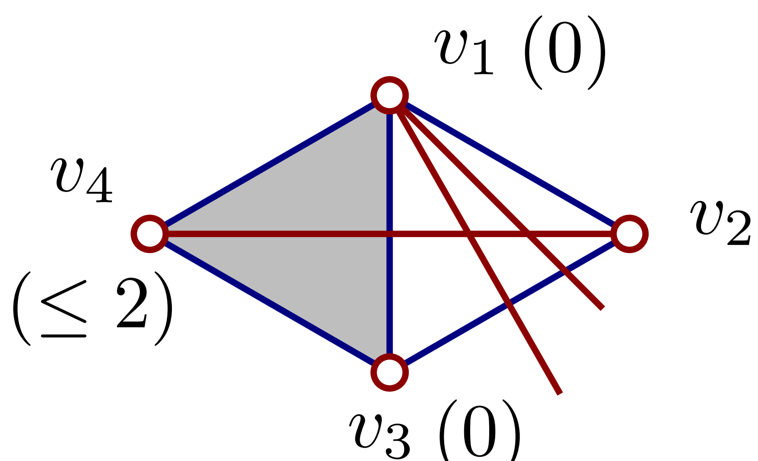


$(3,0,0)$ $(2,1,0)$ $(1,1,1)$

$(2,1,0)$



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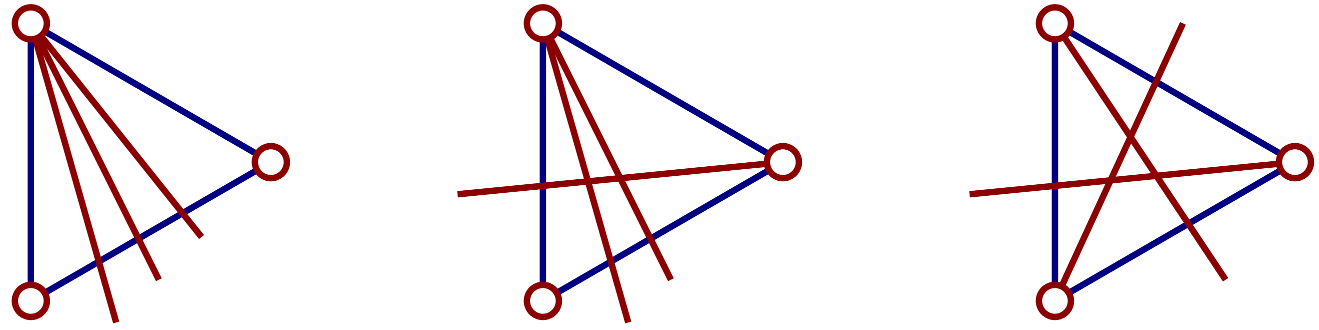
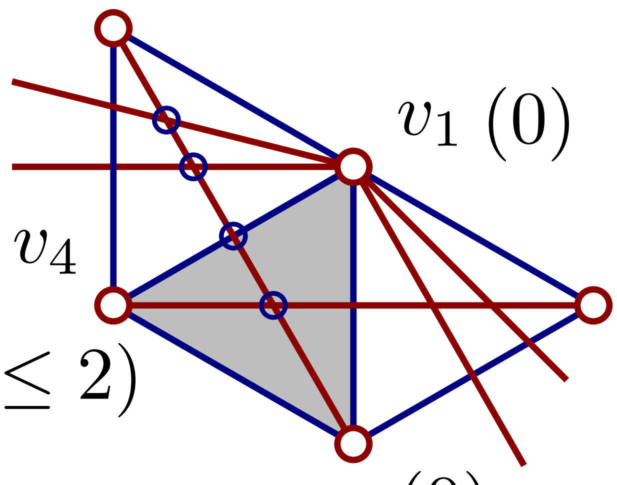


$(3, 0, 0)$ $(2, 1, 0)$ $(1, 1, 1)$

$(2, 1, 0)$



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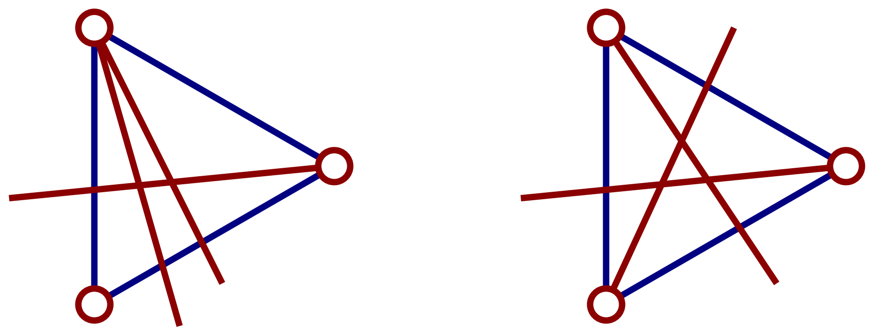

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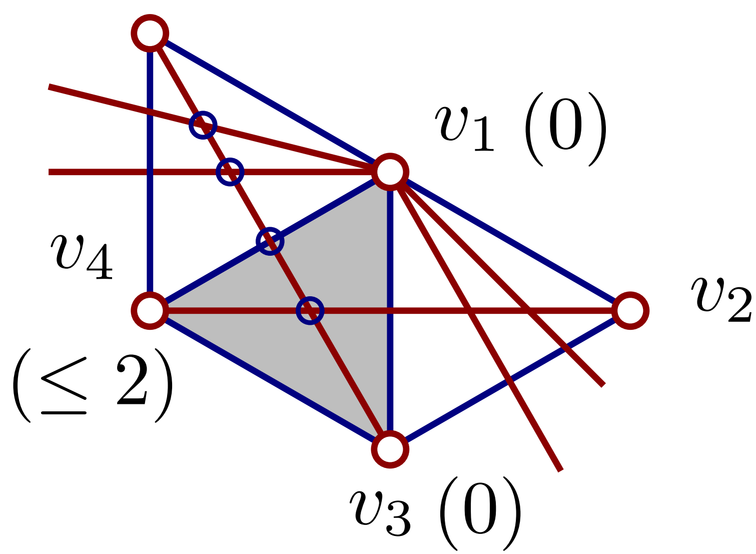
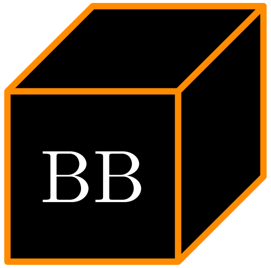



- other cases $\rightarrow$ similarly

# Upper Bound

Maximal plane subgraph $G_p$

- triangular faces
- $e_p = 3n - 6$



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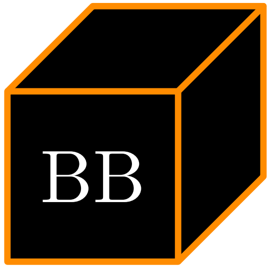


- 11 lemmas
  - connectivity of $G_p$
  - properties of sticks and middle parts

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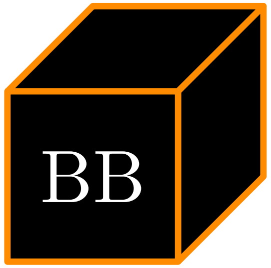


- 11 lemmas
  - connectivity of $G_p$
  - properties of sticks and middle parts

- assumptions
  - $G$ is optimal
  - $G_p$ is maximal
  - $\Gamma(G)$ has minimum number of crossings

# Open Problems

- Characterization of optimal 3-planar graphs
- Tight upper bound for simple 3-planar
- Is $G_p$ fully triangulated for $k = 4$?

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Thank you