On the size of planarly connected crossing graphs

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joint work with
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Planarly connected crossings

- A graph G is **planar** if it can be drawn in the plane without crossing edges.
- **Hanani-Tutte** $\Rightarrow$ If G can be drawn such that every pair of crossing edges shares an endpoint, then G is planar.

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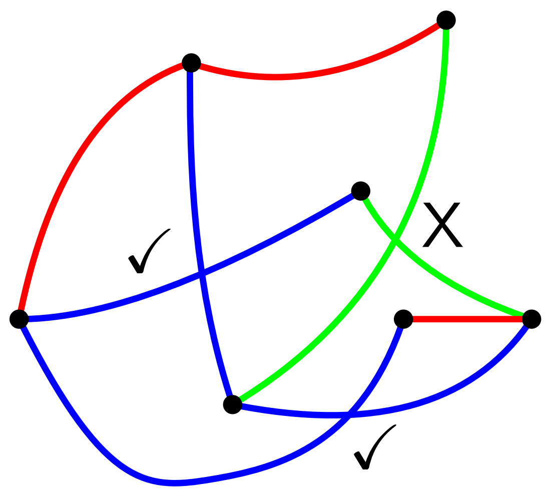
- Two independent and crossing edges are **planarly connected** if there is a crossing-free (planar) edge that connects two of their endpoints.



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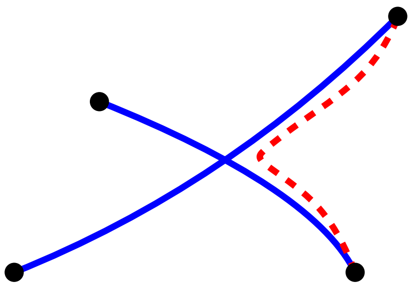

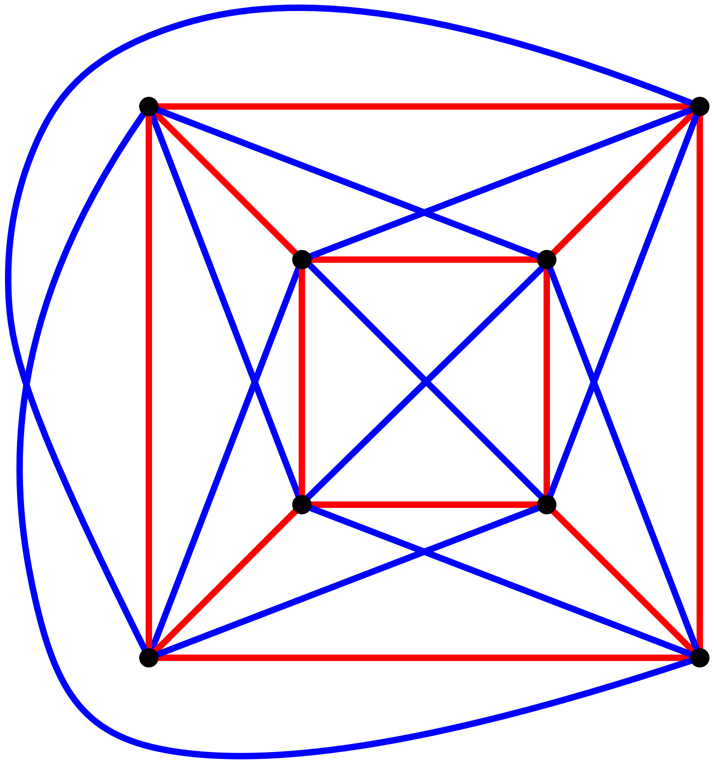

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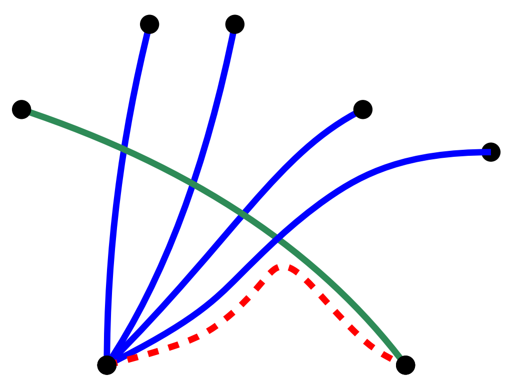

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- Q: Are PCC graphs sparse (have $O(n)$ edges)?

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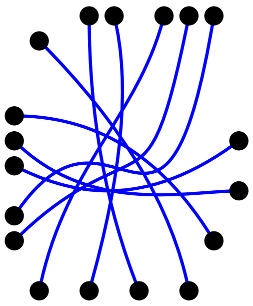
- Conjecture 1 [Pach '91]: k-*quasi-planar* graphs are sparse.
  - Claim: G is 9-quasi-planar.
    - Suppose G has 9 pairwise crossing edges

    $\Rightarrow \binom{9}{2} = 36$ planar edges between their 18 endpoints
    - The induced outerplanar graph has at most $2 \cdot 18 - 3 = 33$ edges



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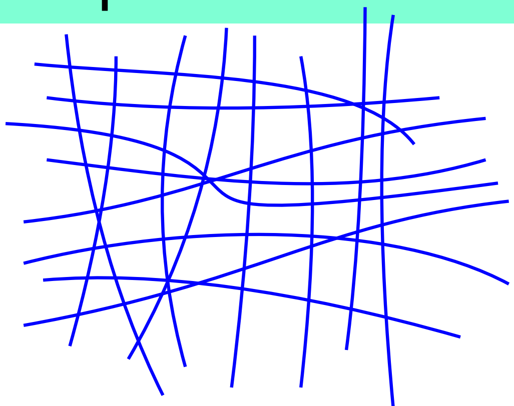
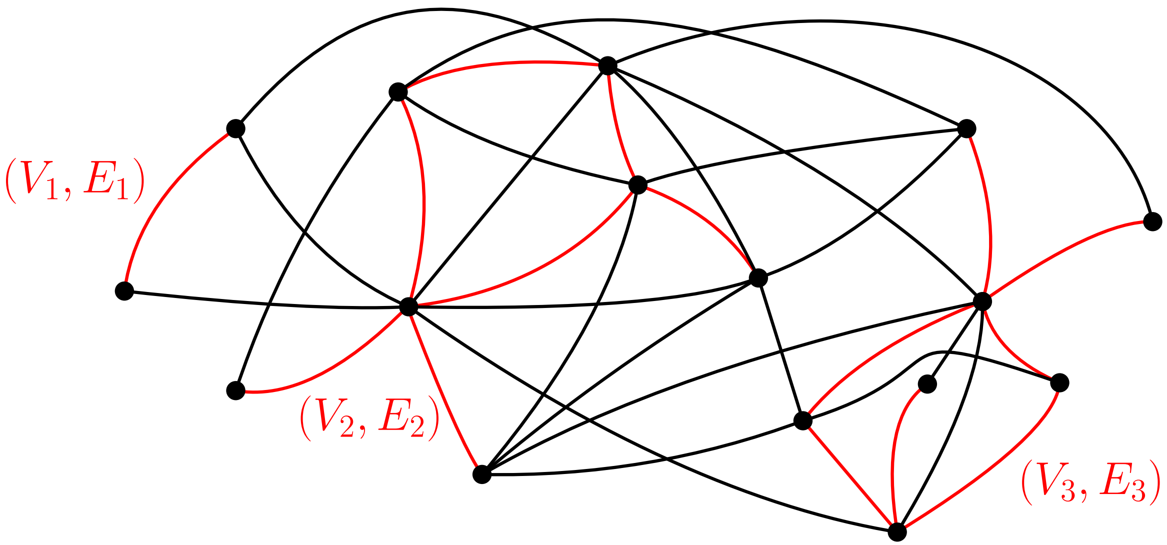
  - Claim: G has no such 8-grid.



PCC graphs are sparse – proof

- $(V_1, E_1), \ldots, (V_k, E_k)$ - connected components of the graph consisting of the planar edges.



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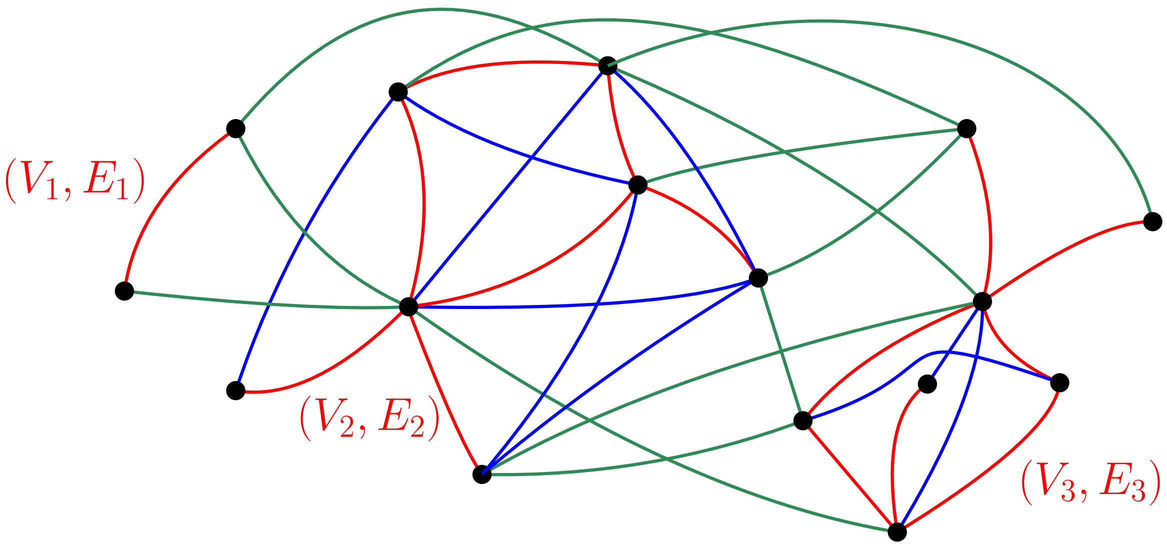
- $(V_1, E_1), \ldots, (V_k, E_k)$ - connected components of the graph consisting of the planar edges.
- Count separately
  - Edges within the same component
  - Edges between different components



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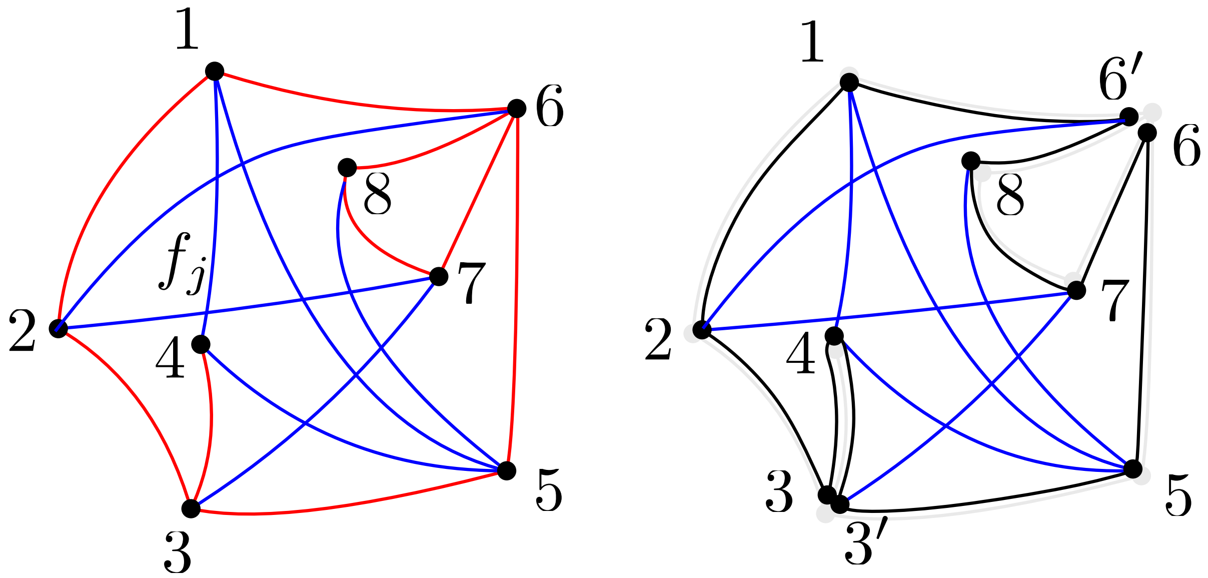

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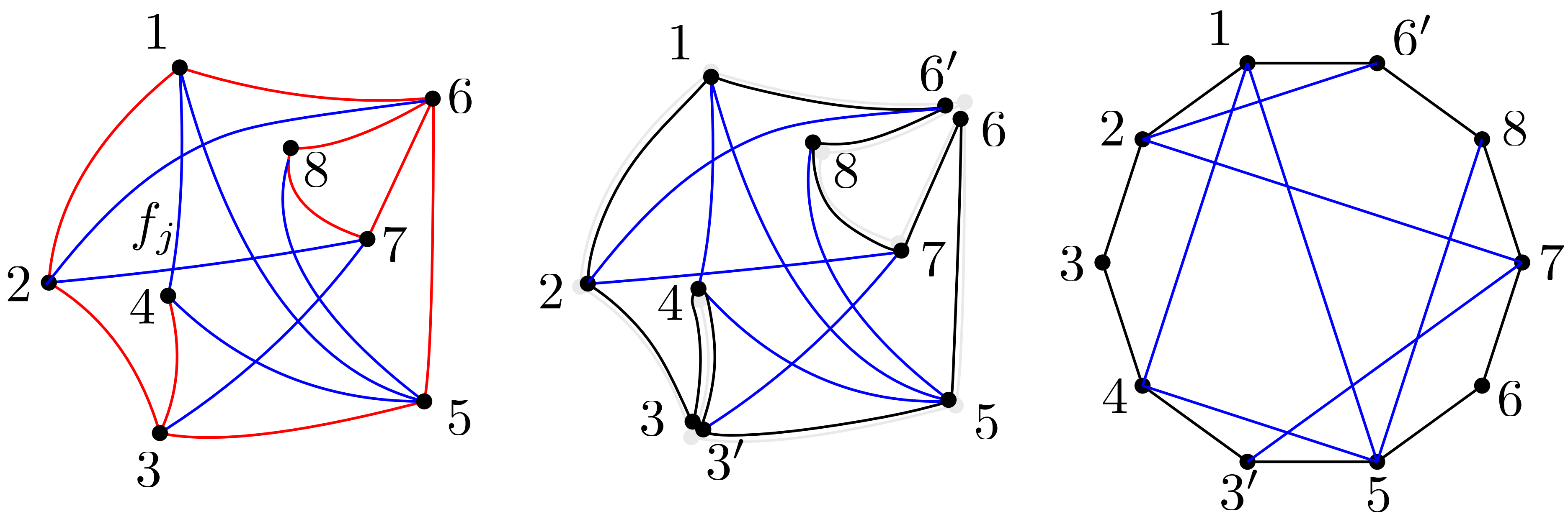

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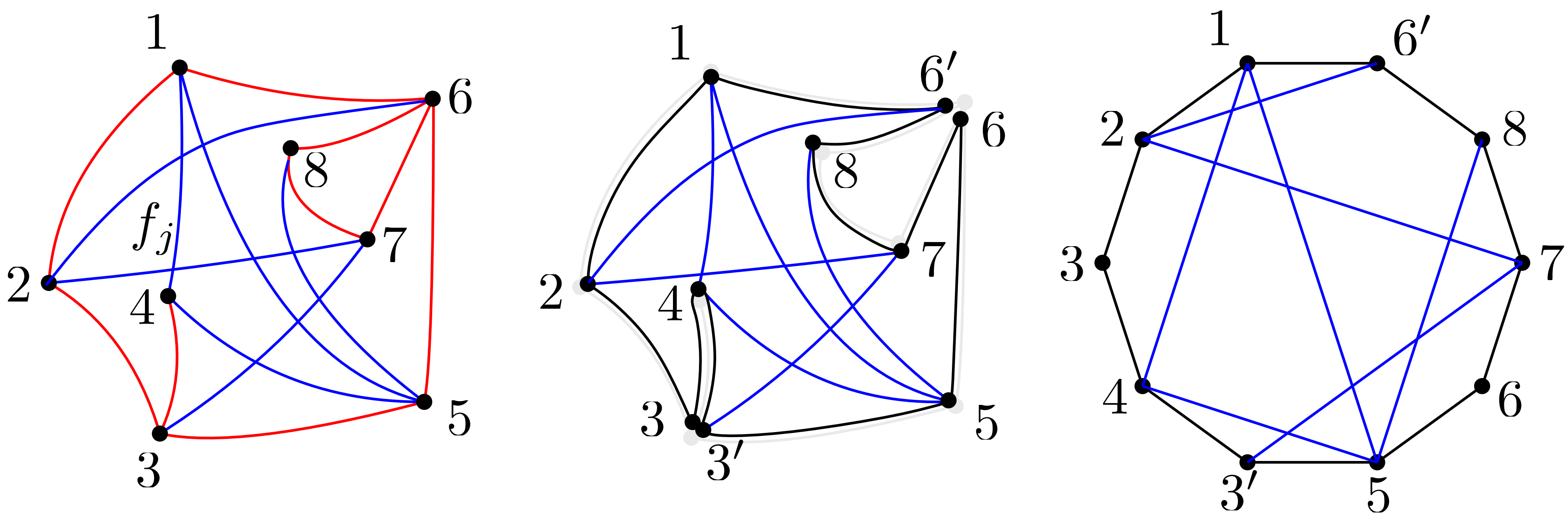

  - The auxiliary graph is 9-quasi-planar and can be realized as a *convex geometric* graph.
  - **Thm** [Capoyleas & Pach '92]: k-quasi-planar convex geometric graphs are sparse.

Edges within a component (V_i, E_i)

- **Lemma 1**: In a face f_j there are $O(|f_j|)$ edges.
- **Corollary 1**: The total number of edges within the k (planar) connected components is at most

$$\sum_{i=1}^{k} O(|E_i|) = O(n)$$

Edges between different components

- $E_{i,j} = \{(v_i, v_j) \in E \mid v_i \in V_i$ and $v_j \in V_j\}$
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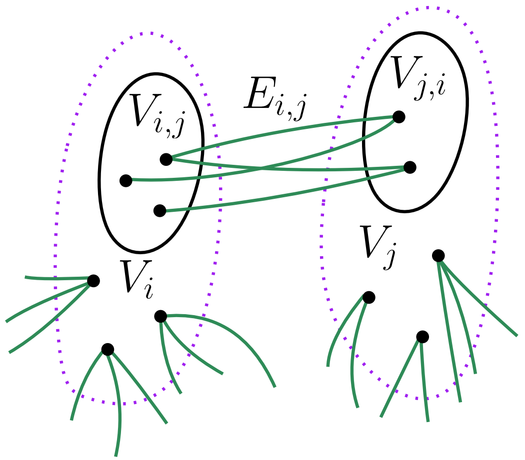

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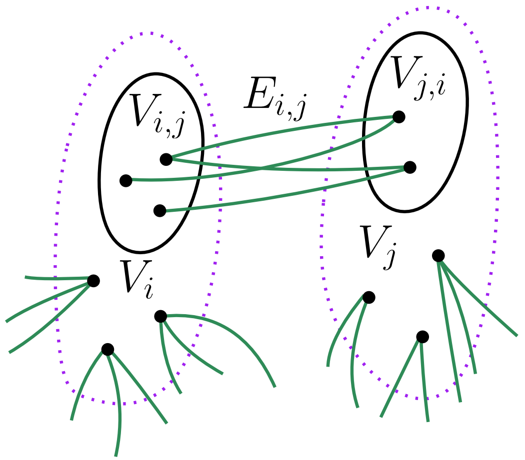

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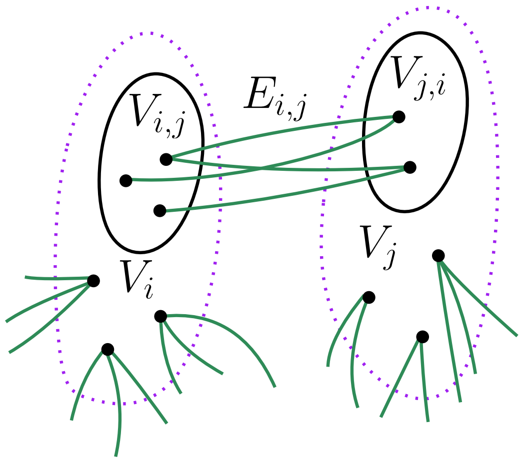

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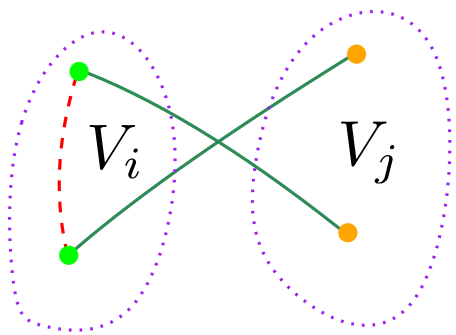
  - Edges connecting a vertex of color c in V_i and a vertex of color c' in V_j cannot cross



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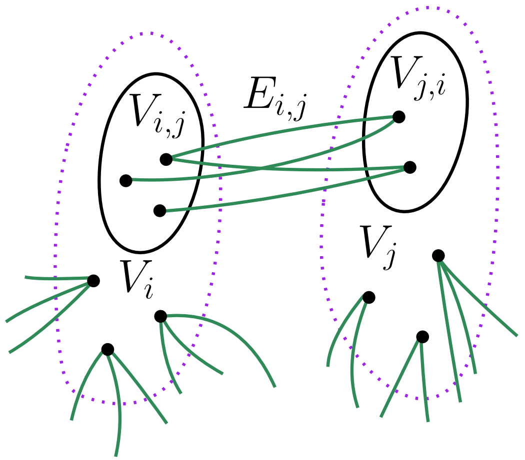

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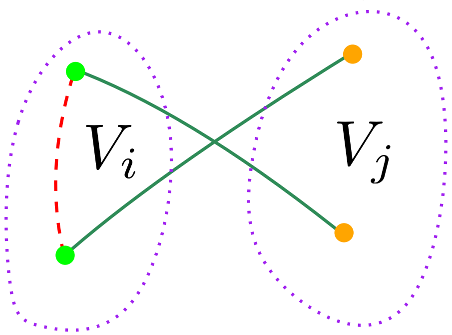

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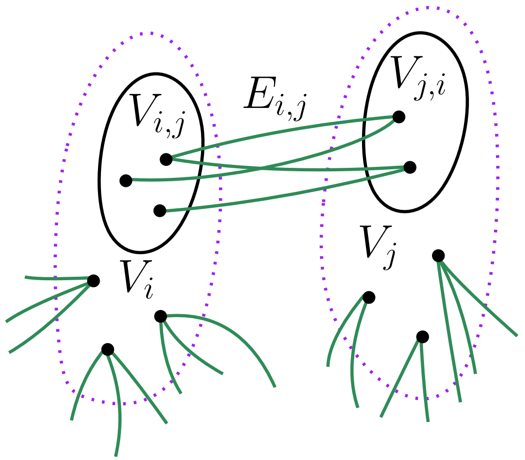

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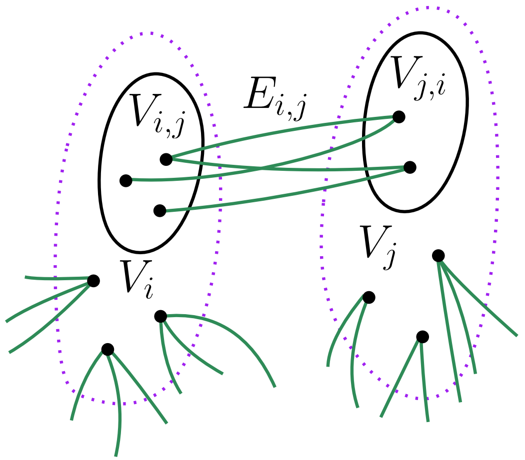

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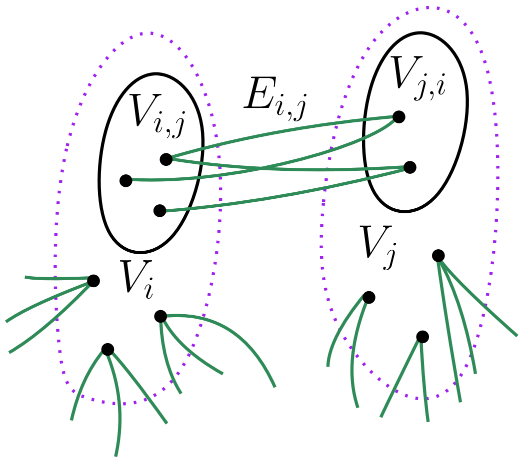

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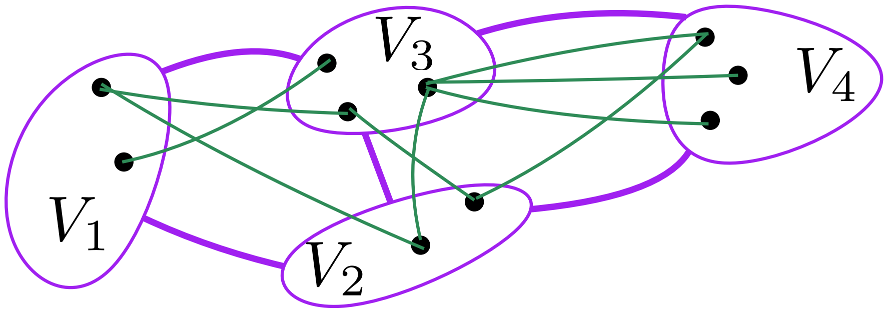
$$\Rightarrow \sum_{i=1}^{k} \sum_{j \neq i} |V_{i,j}| \leq 3(n + 8|E(H)|)$$

The graph H

- $(V_1, E_1), \ldots, (V_k, E_k)$ - connected components of the graph consisting of the planar edges.
- Define a graph H:



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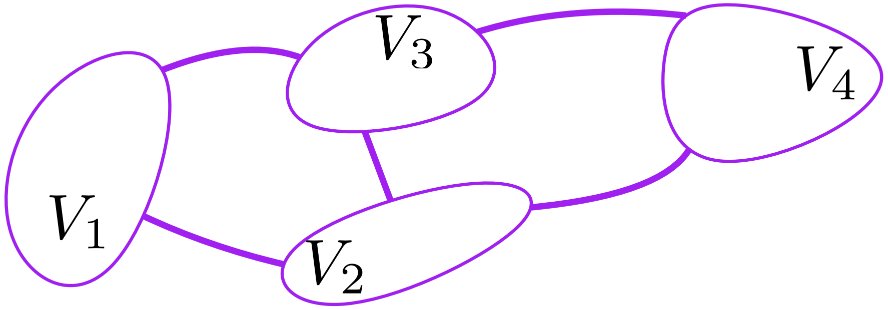

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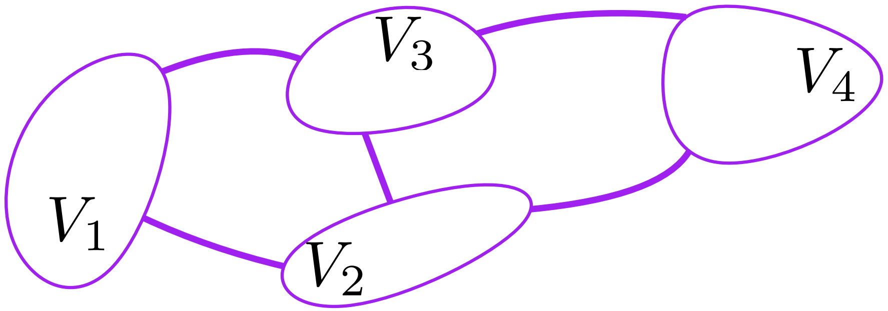

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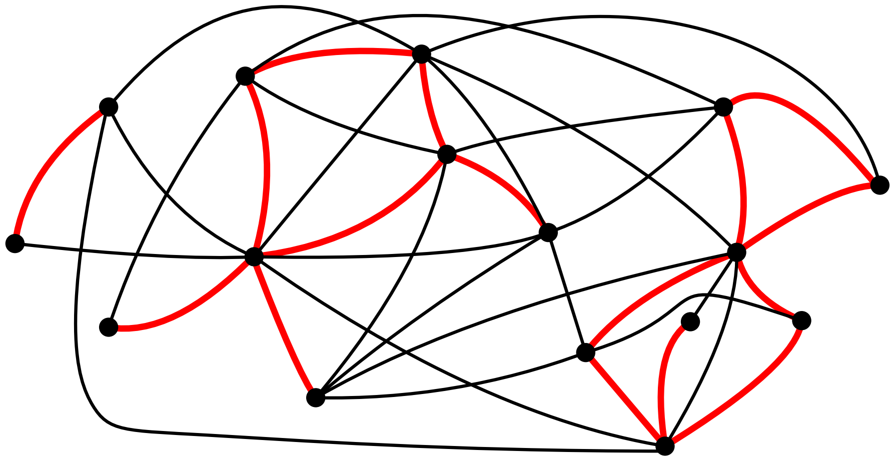

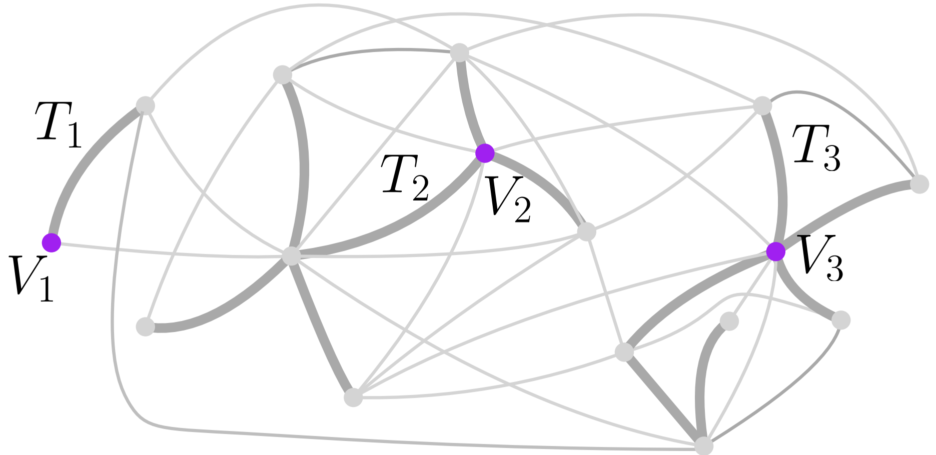

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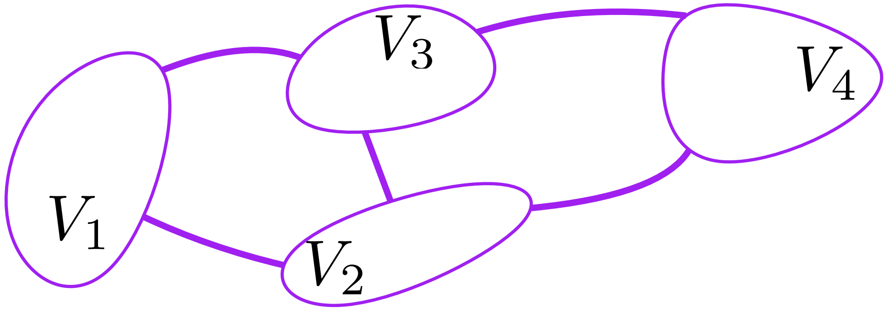

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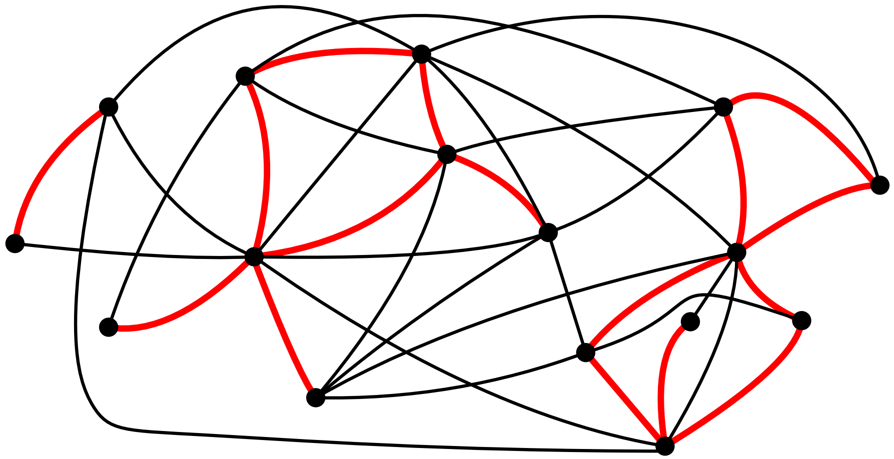

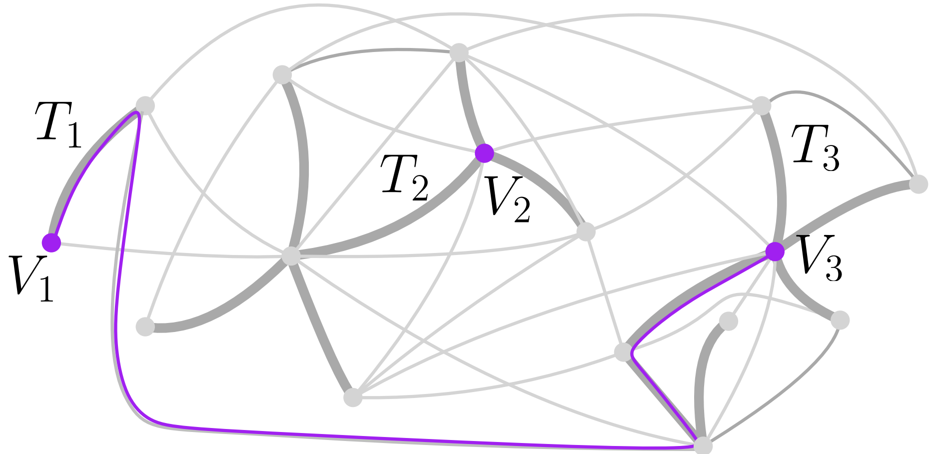

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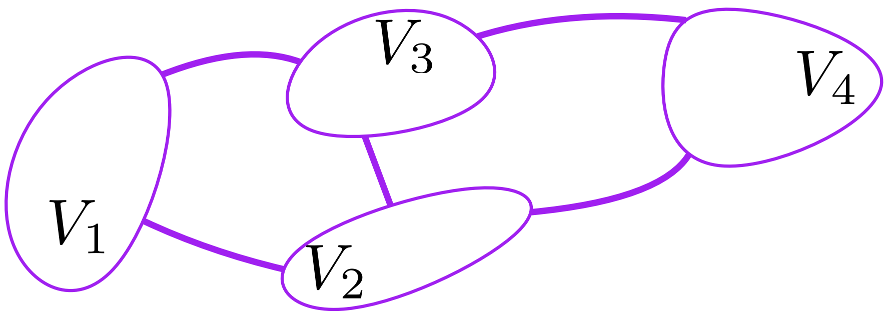

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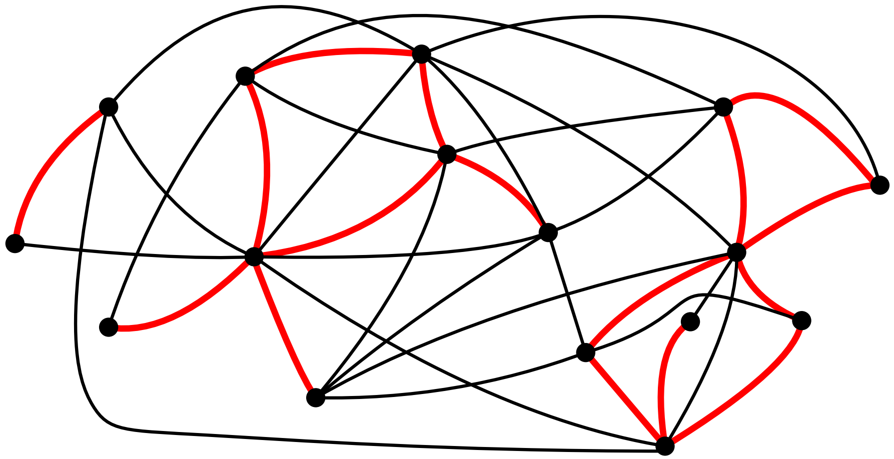

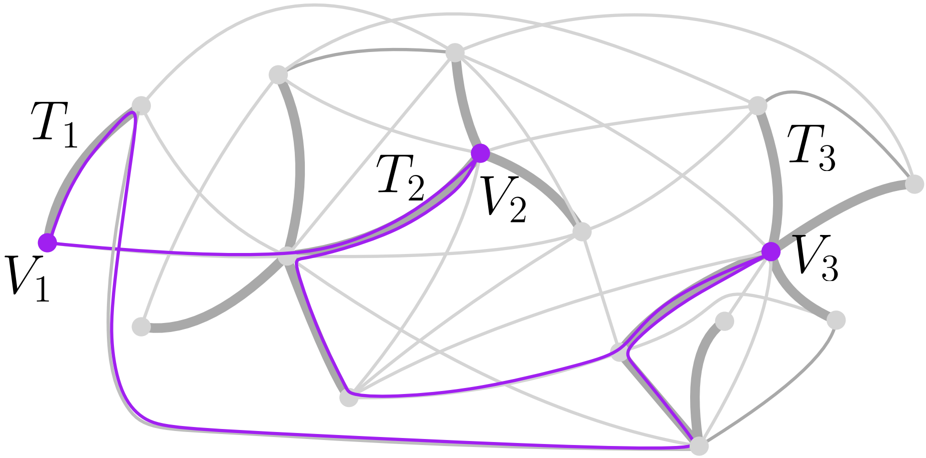

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$\Rightarrow |E| = O(n)$ $\square$

Discussion and open problems

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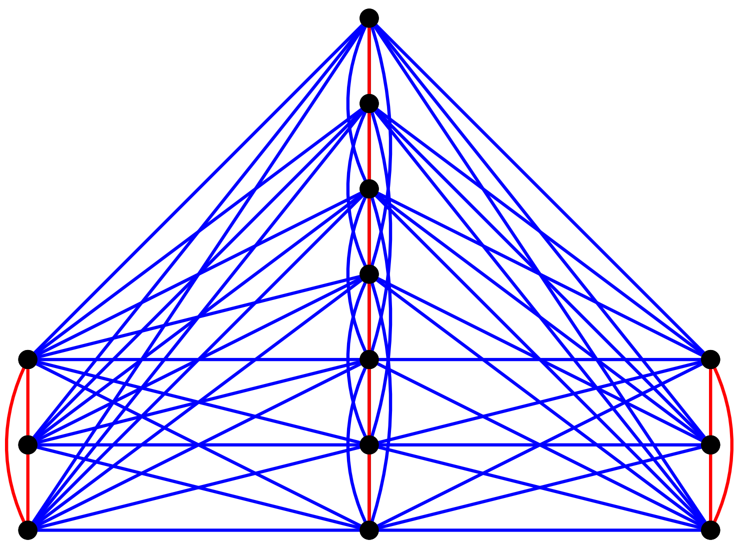
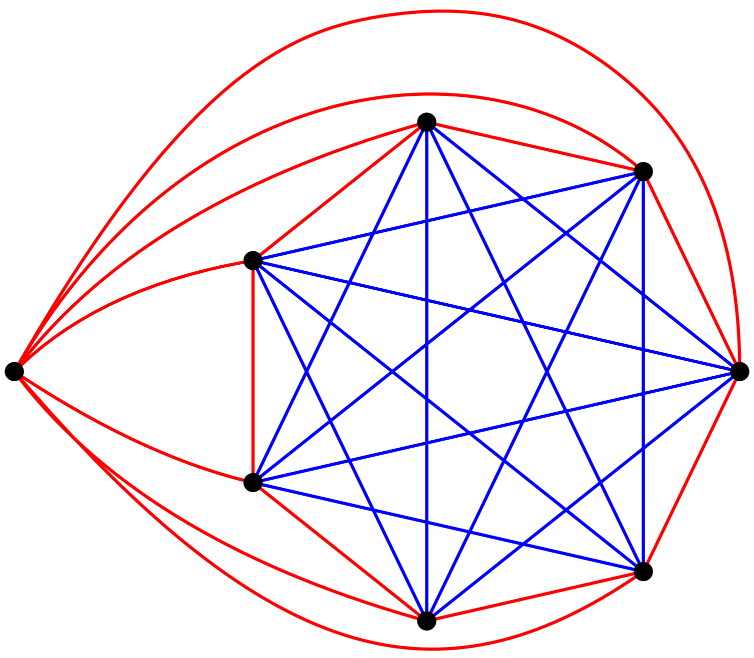
- What if crossing edges may share an endpoint?
- Improve the upper bound on the size of PCC graphs.
  - There are PCC graphs with $9n - O(1)$ edges [G. Tóth]

$\Rightarrow$ Not 1-planar or fan-planar



Discussion and open problems (2)

- If G is 2-PCC — every pair of crossing edges are connected by a planar path of length ≤ 2 — then G may be dense.



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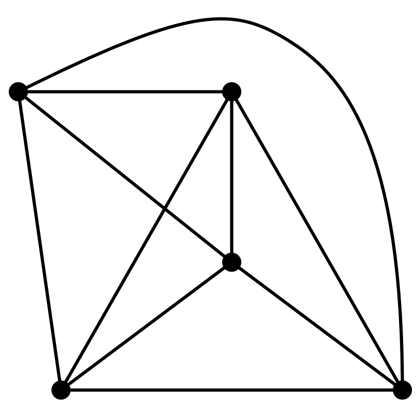

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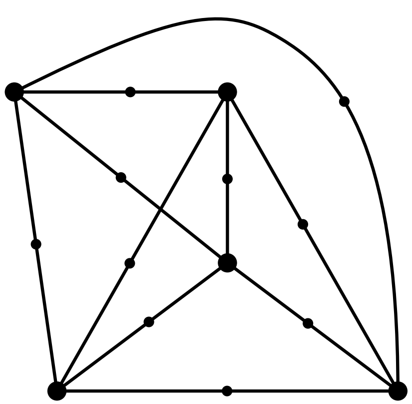

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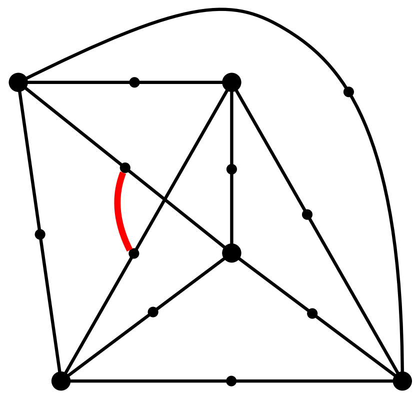

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