Topological Drawings of Complete Bipartite Graphs

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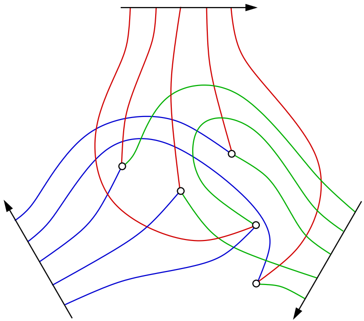


Jean Cardinal
Université Libre de Bruxelles

Stefan Felsner
Technische Universität Berlin

Drawing Model

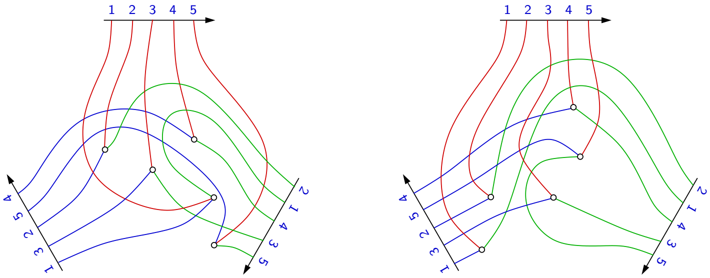
We wish to draw the complete bipartite graph $K_{k,n}$ in the plane in such a way that:

1. vertices are represented by points,
2. edges are continuous curves that connect those points, and do not contain any other vertices than their two endpoints
3. no more than two edges intersect in one point,
4. edges pairwise intersect at most once; in particular, edges incident to the same vertex intersect only at this vertex,
5. the k vertices of one side of the bipartition lie on the outer boundary of the drawing (cyclically as $p_1, \ldots, p_k$).

Properties 1–4 characterize *simple topological drawings* also known as *good drawings*.

Questions and Motivation

- Which sets of rotations $\pi_1, \ldots, \pi_k$ correspond to drawings?
- Structure on set of drawings for given set of rotations?

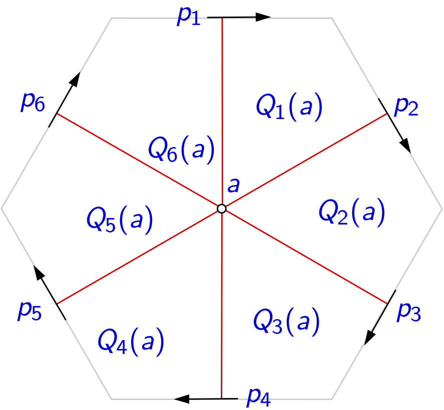


- May have consequences for the bipartite crossing number (Zarankiewicz Conjecture).

Overview

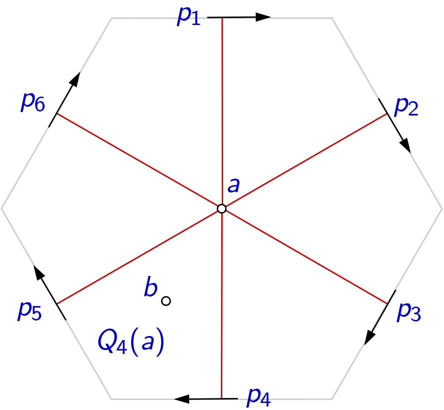
- Uniform rotations.
- Drawings of $K_{2,n}$.
- Drawings of $K_{3,n}$.
- Problems and future work.

Uniform Rotations



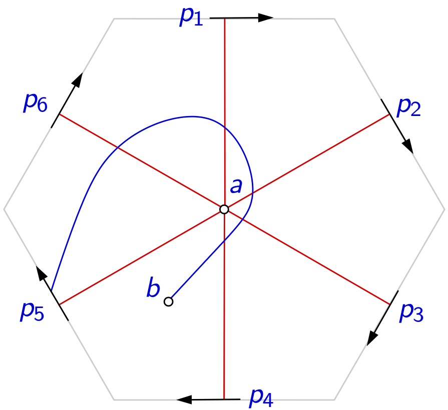
The regions (quadrants) of the inner vertex a.

Uniform Rotations



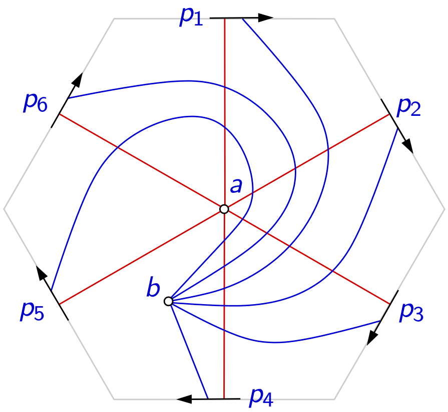
Vertex $b > a$ in quadrant $Q_4(a)$.

Uniform Rotations



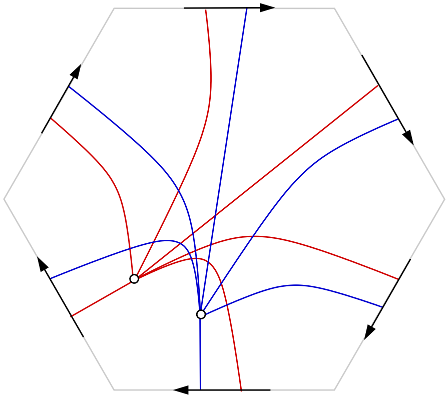
The edge $b \to p_5$ is forced.

Uniform Rotations



All edges $b \to p_i$ are forced.

Uniform Rotations



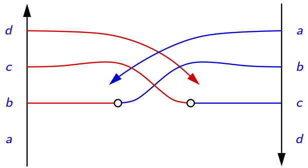
$b \in Q_i(a) \iff a \in Q_i(b)$. Define: $\mathrm{type}(a, b) = i$.

Triple and Quadruple Rule

Lemma. For uniform rotation systems and three vertices $a, b, c \in V$ with $a < b < c$

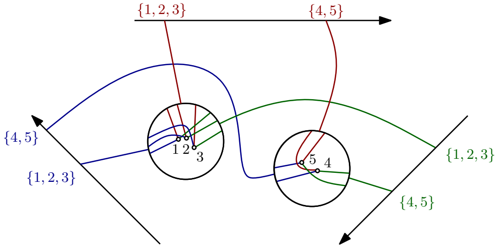
$$\text{type}(a, c) \in \{\text{type}(a, b), \textit{type}(b, c)\}.$$

Lemma. For $a, b, c, d \in V$ with $a < b < c < d$ and any type X:
if $\text{type}(a, c) = \text{type}(b, c) = \text{type}(b, d) = X$ then $\text{type}(a, d) = X$.



Illustrating for the $k = 2$ case of the quadruple rule.

Decomposability and Counting



Theorem. Drawings with uniform rotation system are recursively decomposable.

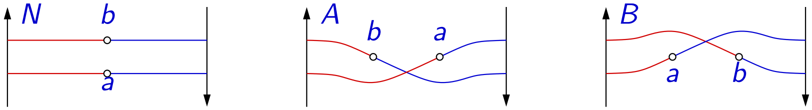
Theorem. Let $T(k, n)$ the number of topological drawings of $K_{k,n}$ with uniform rotation systems

$$T(n+1, k+1) = \sum_{j=0}^{n} \binom{n+j}{2j} C_j \, k^j$$

where C_j is the jth Catalan number.

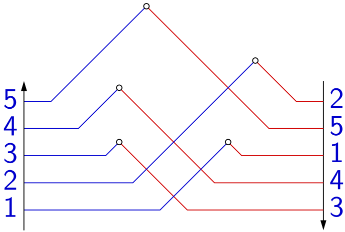
Drawings of $K_{2,n}$

Now we allow arbitrary rotation systems.

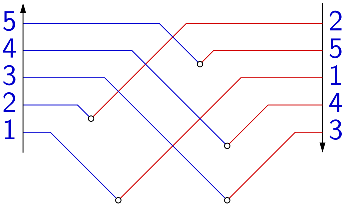


The three types for drawings of $K_{2,2}$.

Two Generic Drawings



1	*N*	*A*	*A*	*N*
	2	*A*	*A*	*A*
		3	*N*	*N*
			4	*N*
				5



1	*N*	*B*	*B*	*N*
	2	*B*	*B*	*B*
		3	*N*	*N*
			4	*N*
				5

Triple and Quadruple Rule

Definition. A triple of types is legal if it corresponds to a drawing of $K_{2,3}$.

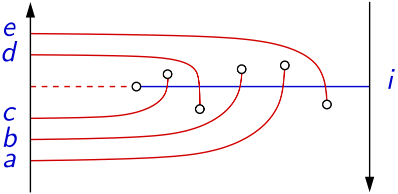
- A triple $\begin{array}{cc|c|c|} \cline{3-4} a & & X & Y \\ \cline{3-4} & b & \multicolumn{1}{c|}{} & Z \\ \cline{4-4} & & & c \end{array}$ with $Y \in \{X, Z\}$ is always legal.

- Additional legal triples: $\begin{array}{cc|c|c|} \cline{3-4} a & & N & A \\ \cline{3-4} & b & \multicolumn{1}{c|}{} & B \\ \cline{4-4} & & & c \end{array}$ and $\begin{array}{cc|c|c|} \cline{3-4} a & & A & B \\ \cline{3-4} & b & \multicolumn{1}{c|}{} & N \\ \cline{4-4} & & & c \end{array}$.

Lemma. Consider four vertices $a, b, c, d \in V$ with $a < b < c < d$.
If $\mathrm{type}(a, b) \neq N$ and $\mathrm{type}(a, c) = \mathrm{type}(b, c) = \mathrm{type}(b, d) = B$ then $\mathrm{type}(a, d) = B$.
If $\mathrm{type}(c, d) \neq N$ and $\mathrm{type}(a, c) = \mathrm{type}(b, c) = \mathrm{type}(b, d) = A$ then $\mathrm{type}(a, d) = A$.

Consistency Theorem

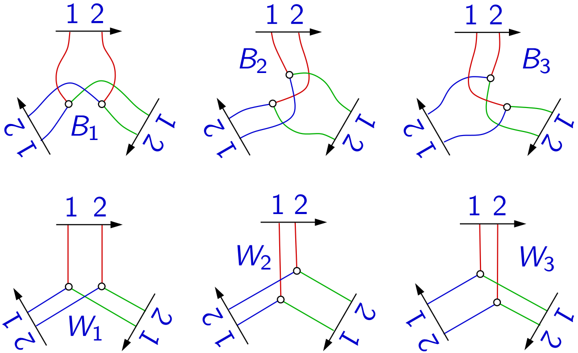
Theorem. Given a type for each pair of vertices in V, there exists a drawing realizing those types if and only if all triples are legal and the quadruple rule is satisfied.



a	A	A	B
b	A	A	B
c	A	B	B
	i	B	B
		d	e

- The quadruple rule allows to sort the crossings of an edge consistently.
- The triple rule allows to combine the local sequences of all vertices into an arrangement of pseudolines.

Drawings of $K_{3,n}$



The six types for drawings of $K_{3,2}$.

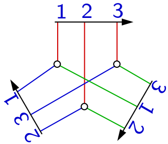
Remark. Types B_i are ineligible for straight line drawings.

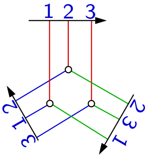
Classification of Drawings of $K_{3,3}$

- All decomposable triples $\begin{array}{|c|c|}\hline X_\alpha & X_\alpha \\ \hline & Y_\beta \\ \cline{2-2}\end{array}$ or $\begin{array}{|c|c|}\hline X_\alpha & Y_\beta \\ \hline & Y_\beta \\ \cline{2-2}\end{array}$.

- Two additional mixed systems:

$\begin{array}{|c|c|}\hline W_3 & W_1 \\ \hline & W_2 \\ \cline{2-2}\end{array}$

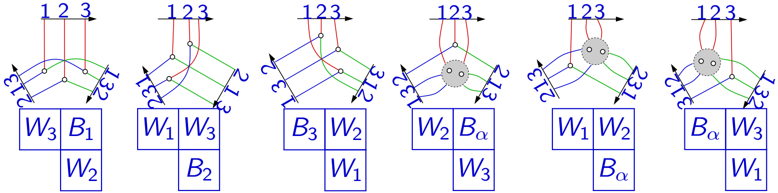




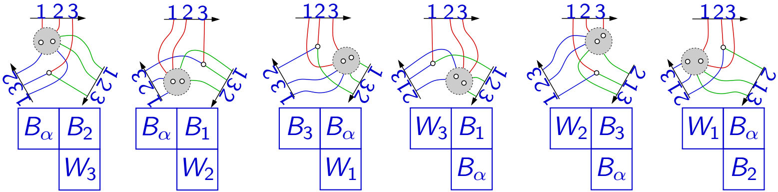
$\begin{array}{|c|c|}\hline W_2 & W_1 \\ \hline & W_3 \\ \cline{2-2}\end{array}$

Classification of Drawings of $K_{3,3}$ — II

Non-decomposable tables with two mixed and one uniform pair:



Non-decomposable tables with one mixed and two uniform pairs:



Consequences

- There are 92 drawings of $K_{3,3}$ (consistent tables). Of these 66 are decomposable and 26 non-decomposable.
- A table is consistent for $K_{3,3}$ if and only if all three projections to tables for $K_{2,3}$ are consistent. (Computer check).
- There are non-realizable systems of rotations.

Consequences

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Example. The system $(\mathrm{id}_4, [4,2,1,3], [2,4,3,1])$ is an infeasible. The table of types for the given permutations is

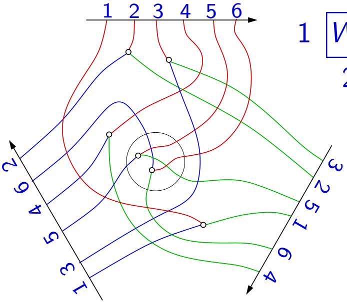
1	W_1	W_3	W_1
	2	B_α	W_2
		3	W_1
			4

The subtable corresponding to $\{1,2,3\}$ implies $\alpha = 2$.

The subtable corresponding to $\{2,3,4\}$ implies $\alpha = 3$.

An Example with 3 Realizations

1	W_2	W_2		W_2	
	2	W_1	W_3	W_3	W_3
		3			
			4	W_1	W_2
				5	
					6



1	W_2	W_2	B_3	W_2	B_3
	2	W_1	W_3	W_3	W_3
		3	B_2	B_2	B_2
			4	W_1	W_2
				5	B_α
					6

The type of mixed pairs is given by the rotations.
Consistency forces the type of most uniform pairs.

The Quadruple Rule

1	W_1	B_1	B_1
	2	B_1	B_2
		3	B_2
			4

2	A	B	A
	1	B	B
		3	B
			4

The table on the left is consistent on all triples.
The projection to green-blue (resorted according to $\pi_3 = (2, 1, 3, 4)$) reveals a bad quadruple.

Definition. T is *consistent on quadruples* if for any four vertices a, b, c, d and $i \in \{1, 2, 3\}$ the projection to π_{i-1} and π_{i+1} satisfies the qudruple rule for $K_{2,n}$.

The Consistency Theorem

Theorem. Given a type for each pair of vertices in V, there exists a drawing realizing those types if and only if all triples and quadruples are consistent.

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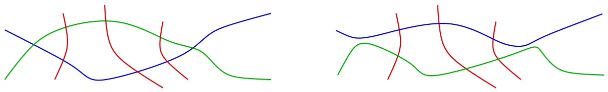
Idea for the proof

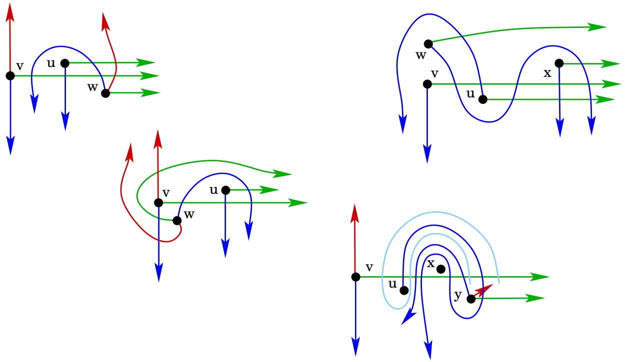
- Produce drawings realizing the red-green and the red-blue projections.
- Superimpose the drawings such that the red stars coincide and the rotations at the inner vertices are red-green-blue in clockwise order.
- Get rid of empty lenses (i.e., lenses that do not contain a vertex).



The Hard Part of the Proof

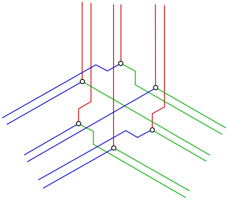
Proposition. There is no lens that contains a vertex.

Cases.



Linear and Pseudolinear Drawings

- Linear and pseudolinear drawings only exist for mixed systems, i.e., all pairs have type B_α for $\alpha \in 1, 2, 3$.
- If (π_1, π_2, π_3) is a mixed system of rotations, then there is a drawing realizing the system.
- There are pseudolinear drawings with no linear realization.

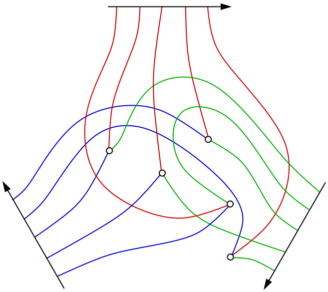


- Testing strechability is easy because we only allow 3 directions.

Open Problems and Future Work

- Structure and enumeration for the set of drawings of $K_{2,n}$ with given rotations (id, π).
- Deciding existence of a drawing of $K_{3,n}$ with given rotations $(\mathrm{id}, \pi_2, \pi_3)$.
- Extensions to drawings of $K_{k,n}$. (It should hold that a table is consistent if and only if all projections are consistent. It might be enough that projections to (π_i, π_{i+1}) are consistent.)
- A table with B and W positions prescribed (no indices). It is NP-complete to decide whether there are corresponding permutations $(\mathrm{id}, \pi_2, \pi_3)$.
- Extensions to drawings without the condition that one color-class is on the boundary of the drawing.

The End



Thank You