A Direct Proof of the Strong Hanani–Tutte Theorem on the Projective Plane

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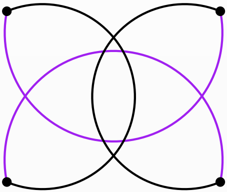
Definition

A drawing D of a graph G on a surface S is called a ***Hanani–Tutte*** *drawing if any two non-incident edges cross an even number of times in D.*



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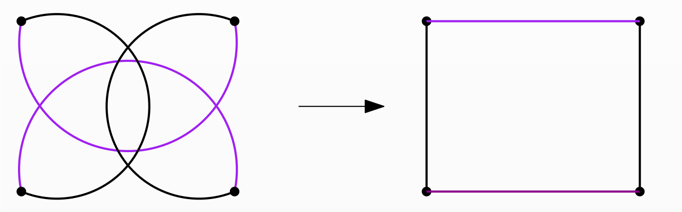
Theorem (Hanani–Tutte)

A graph G is planar if and only if it has a Hanani–Tutte drawing in the plane.



Conjecture

For every (closed) surface S a graph G is embeddable into S if and only if it has a Hanani–Tutte drawing on S.

These pictures are taken from Wikipedia

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  - [Tutte '70] proved the case of S^2 using Kuratowski's theorem.
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Introduction

The Hanani–Tutte conjecture

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  - [Pelsmajer, Schaefer, Stasi '09] proved the case of $\mathbb{R}P^2$ using the forbidden minors.

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- The approach via forbidden minors is not usable for higher-genus surfaces.
  - The exact lists of the forbidden minors are not know except for S^2 and $\mathbb{R}P^2$.
  - Already for the torus there are thousands; a complete list is not known.
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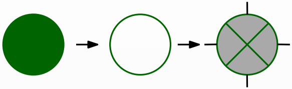
  - Already for the torus there are thousands; a complete list is not known.
  - Their number is increasing in the genus.
- **Our contribution**: we provide a constructive proof of the case on $\mathbb{R}P^2$.

- We represent $\mathbb{R}P^2$ as S^2 with a **crosscap** attached to it
  - A crosscap is a topological disk with its interior removed and the opposite points on its boundary identified
  - We draw it as $\otimes$



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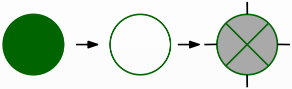

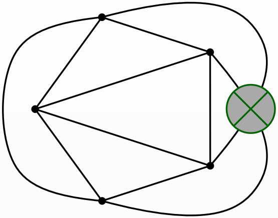

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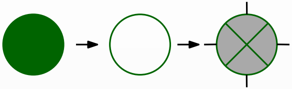
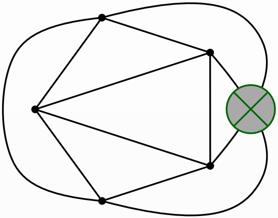


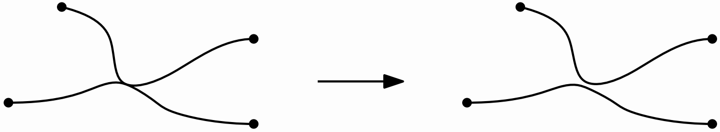


Definition

*Let D be a drawing of a graph G on $\mathbb{R}P^2$. We say that an edge e is **nontrivial** in D if e crosses the crosscap an odd number of times; otherwise e is **trivial**. We say that a walk in G is **nontrivial** in D if it crosses the crosscap an odd number of times.*

We put the standard general position assumptions on the drawings:

- Whenever two edges meet at a point, they cross there transversally



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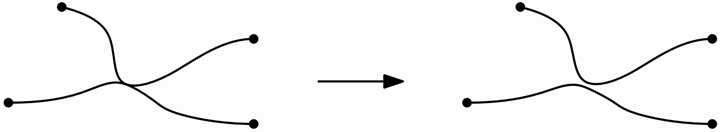

Definition

*We say that an edge e is **even** in a drawing if it crosses every other edge an even number of times.*

Definition

*A curve is **simple** if it does not intersect itself.*

Our proof

- On the top level, our strategy is the same as in [Pelsmajer, Schaefer, Štefankovič '07]
- Their inductive redrawing procedure has to be replaced by something much more involved
- The main induction is more complicated

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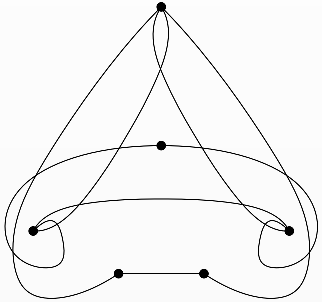

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  1. Start with a Hanani–Tutte drawing
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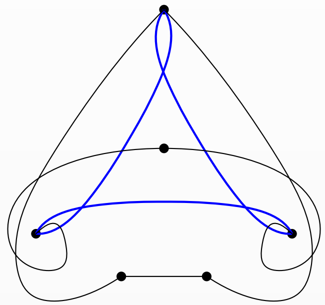

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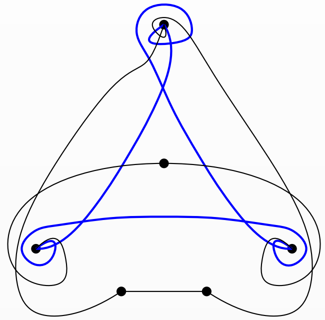

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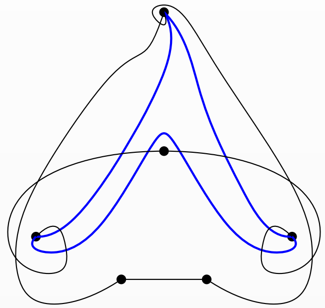

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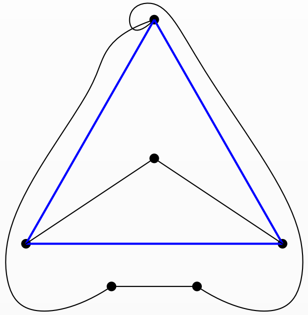

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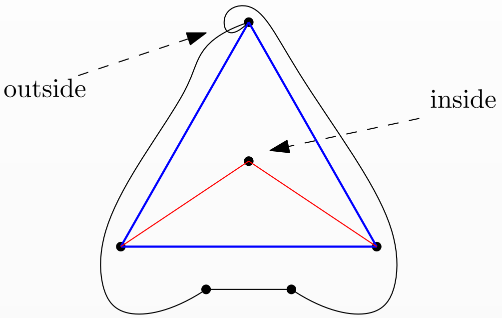
  6. Cycle C splits the graph into two parts; redraw them inductively



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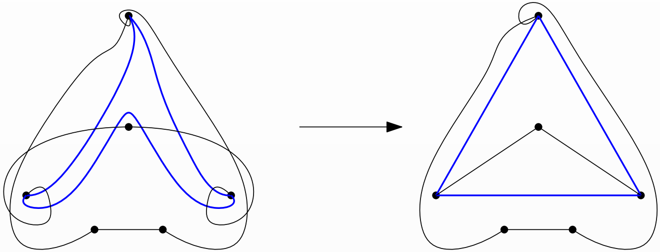

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Theorem ([Pelsmajer, Schaefer, Štefankovič '07])

If D is a drawing of a graph G in S^2, and E_0 is the set of even edges in D, then G can be drawn in S^2 so that no edge in E_0 is involved in an intersection and there are no new pairs of edges that intersect an odd number of times.

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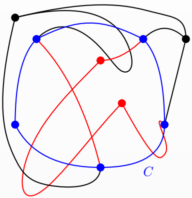
- It ensures that C can be made free of crossings and kept such during the induction
- **The main obstacle**: the theorem is simply not true for other surfaces [Pelsmajer, Schaefer, Štefankovič '07b]
- We provide its suitable replacement on $\mathbb{R}P^2$

- When the edges of C are even and trivial and C is drawn as a simple cycle $\Longrightarrow$ it separates the graph into two parts called the **inside** and the **outside**



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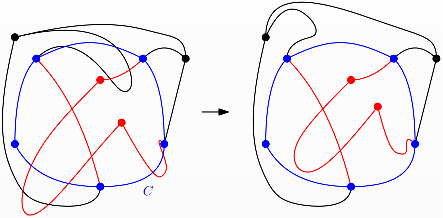

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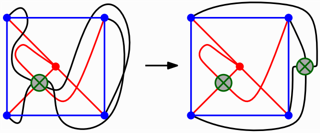
- But both the inside and the outside may use the crosscap on $\mathbb{R}P^2$...



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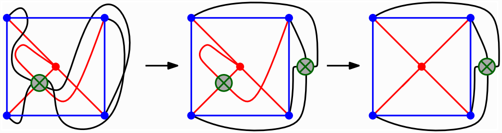


- However, we show that at least one of the sides can be *always* redrawn without using the crosscap (yielding a Hanani-Tutte drawing)

In order to show that the inside or the outside of C can be redrawn without using the crosscap we investigate patterns among nontrivial walks inside and outside with the endpoints on C.

To this end we use mainly

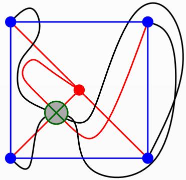
Proposition (Intersection form on the projective plane)

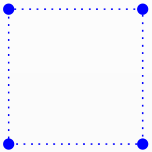
Two nontrivial cycles have to cross an odd number of times on $\mathbb{R}P^2$.

... and a technical tool that we call an **arrow graph**.

The **arrow graph** is a multigraph A such that:

- The vertices of the arrows are the vertices of C





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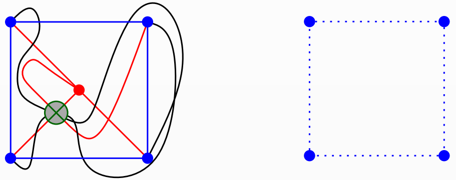

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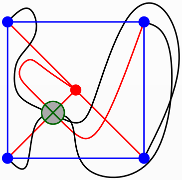

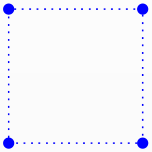

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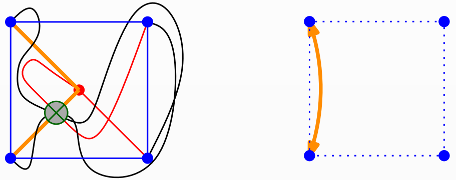

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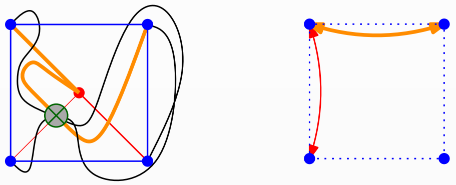

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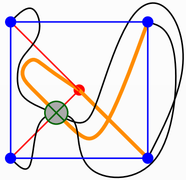

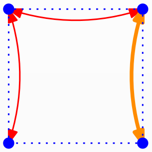

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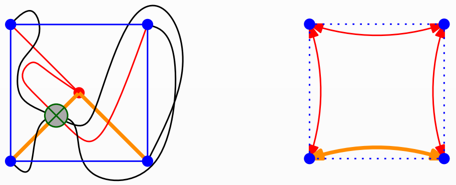

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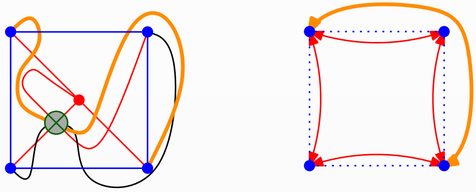

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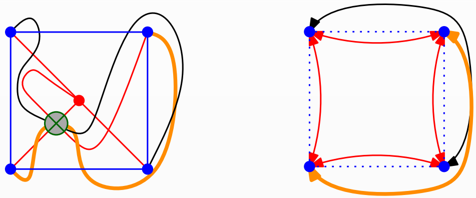

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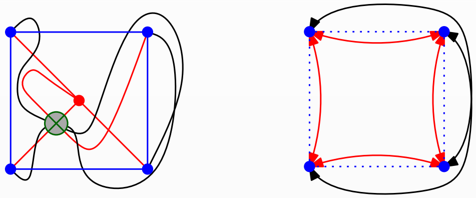
- The arrow graph naturally splits into two subgraphs - the inside arrows (red) and the outside arrows (black)



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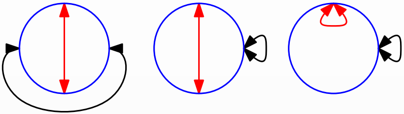
- This means that the following configurations of arrows are forbidden:



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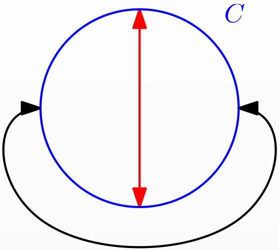

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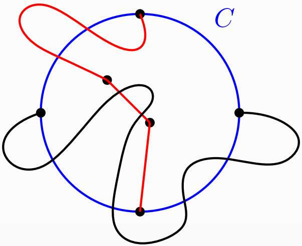

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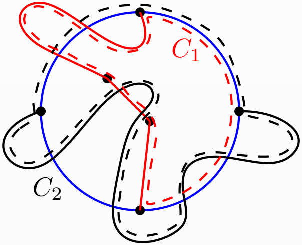

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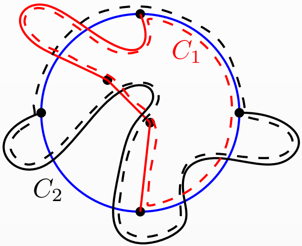

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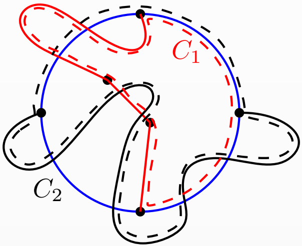
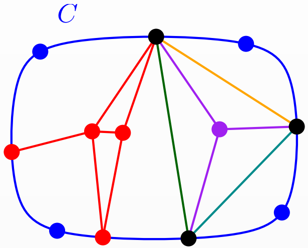
    - But C_1 and C_2 have to cross oddly $\Longrightarrow$ a contradiction!



Definition

A ***bridge*** *is a subgraph that is either an edge not in C but with both endpoints on C or a connected component of $G \setminus V(C)$ together with all edges with one endpoint in that component and the other endpoint in C.*



By the definition of the arrows, a walk defining an arrow is always fully contained in one of the bridges.

Property of the arrow graph

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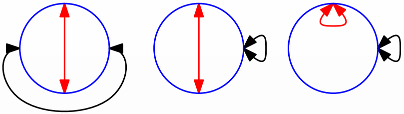
Forbidden configuration(s)



Our proof

The arrows - forbidden configurations

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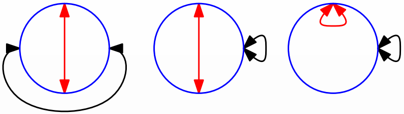
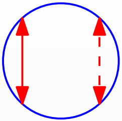


Lemma

The endpoints of two disjoint inside (or outside) arrows induced by different inside (outside) bridges have to interleave.



Our proof

The arrows - forbidden configurations

Property of the arrow graph | Forbidden configuration(s)

Lemma

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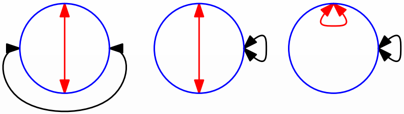

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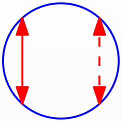
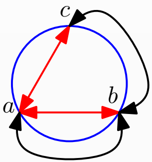


Lemma

There are no three distinct vertices a, b and c on C such that the arrows $\overline{ab}$ a $\overline{ac}$ are induced by an inside component and $\overline{ab}$ and $\overline{bc}$ are induced by an outside component.



We can use the forbidden configurations from the previous slide to show that one of the following configurations of arrows has to appear either inside or outside:

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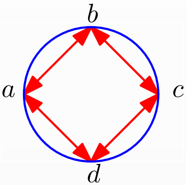
- A **fan** = there is a vertex common to all arrows on one of the sides
- A **square** = there are four vertices a, b, c and d on C in this order and four arrows $\overline{ab}, \overline{bc}, \overline{cd}$ and $\overline{ad}$ inside (outside). Moreover, there is only one nontrivial bridge inside (outside)



Our proof

The arrows - redrawable configurations

We can use the forbidden configurations from the previous slide to show that one of the following configurations of arrows has to appear either inside or outside:

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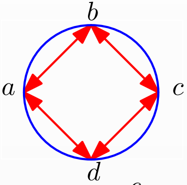
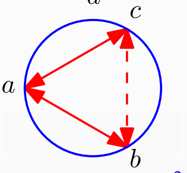
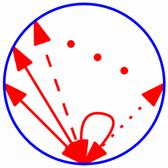


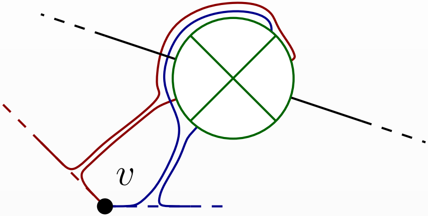
- A **split triangle** = there are three vertices a, b and c on C such that only the arrows $\overline{ab}, \overline{bc}$ and $\overline{ac}$ can be present on both sides. Moreover, on one of the sides, every nontrivial bridge induces either a single arrow or the arrows $\overline{ab}$ and $\overline{ac}$





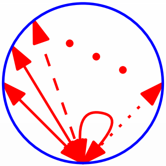
In order to redraw the fan:

- We use a **vertex-crosscap switch** = an opperation that preserves Hanani–Tutte drawings and swaps triviality/nontriviality of all edges incident to a chosen vertex v



Our proof

The arrows - redrawing the fan



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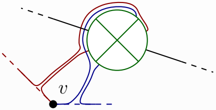
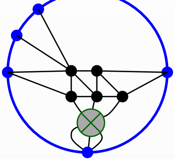
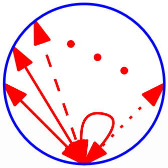




- By vertex-crosscap switches change the drawing so that all nontrivial edges become incident to the common endpoint of the arrows forming the fan

In order to redraw the fan:



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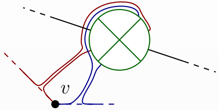
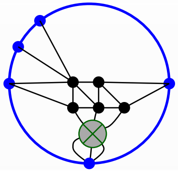
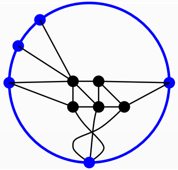
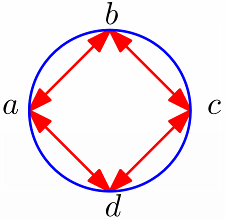


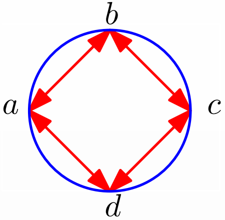
- By vertex-crosscap switches change the drawing so that all nontrivial edges become incident to the common endpoint of the arrows forming the fan
- Now simply remove the crosscap and draw the edges there with a crossing
- These crossings do not matter since the edges are incident $\Longrightarrow$ we still have a Hanani–Tutte drawing





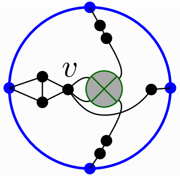


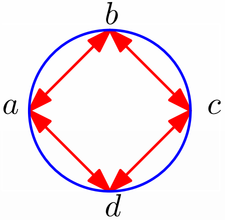
In order to redraw the square:



In order to redraw the square:

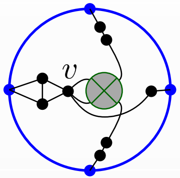
- Show that the nontrivial component inducing the arrows of the square contains a cut vertex v

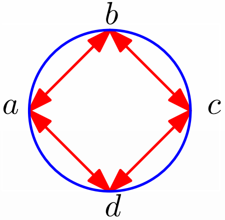




In order to redraw the square:

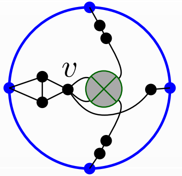
- Show that the nontrivial component inducing the arrows of the square contains a cut vertex v
- By vertex-crosscap switches ensure that all nontrivial edges become incident to v

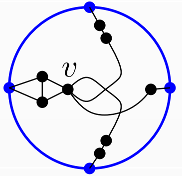




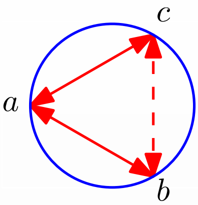
In order to redraw the square:

- Show that the nontrivial component inducing the arrows of the square contains a cut vertex *v*
- By vertex-crosscap switches ensure that all nontrivial edges become incident to *v*
- Now remove the crosscap and draw the edges there with a crossing



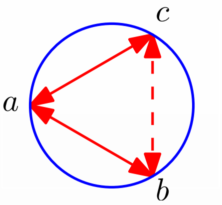


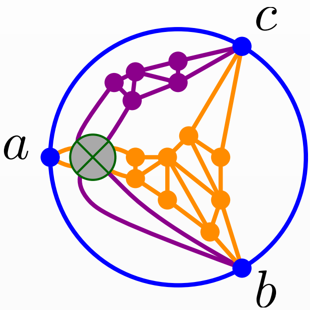
In order to redraw the split triangle:



In order to redraw the split triangle:

- One has to work a bit more...





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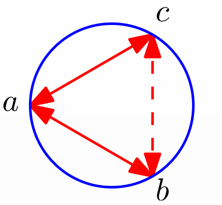

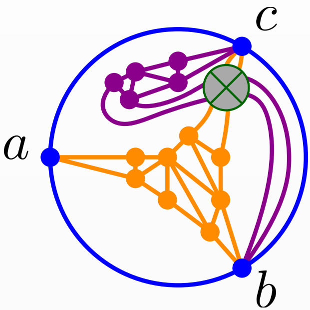

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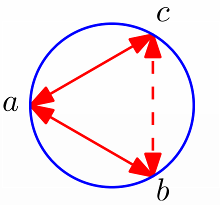

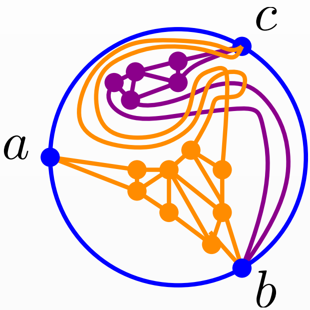

- The tools we use can be generalized to other surfaces, at least to non-orientable ones
- But the number of redrawable configurations seems to increase with the genus
- If the Hanani–Tutte conjecture is true, we would need either to develop better redrawing tools or to further restrict the redrawable configurations in order to prove the conjecture
- However, if it is not true, then our tools could reveal an appropriate structure to disprove the conjecture

Thank you for your attention!
Questions?