Hanani–Tutte for radial drawings

Radoslav Fulek,IST Austria

IST AUSTRIA
Institute of Science and Technology

(Marcus Schaefer, De Paul Chicago and Michael Pelsmajer, IIT Chicago)

Hanani–Tutte theorem

Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

Shapiro (1957), Wu (1970s): Given a graph $G = (V, E)$. We can test in a polynomial time whether G can be drawn so that **no two non-adjacent** edges of G cross an **odd** number of times.

Hanani–Tutte theorem

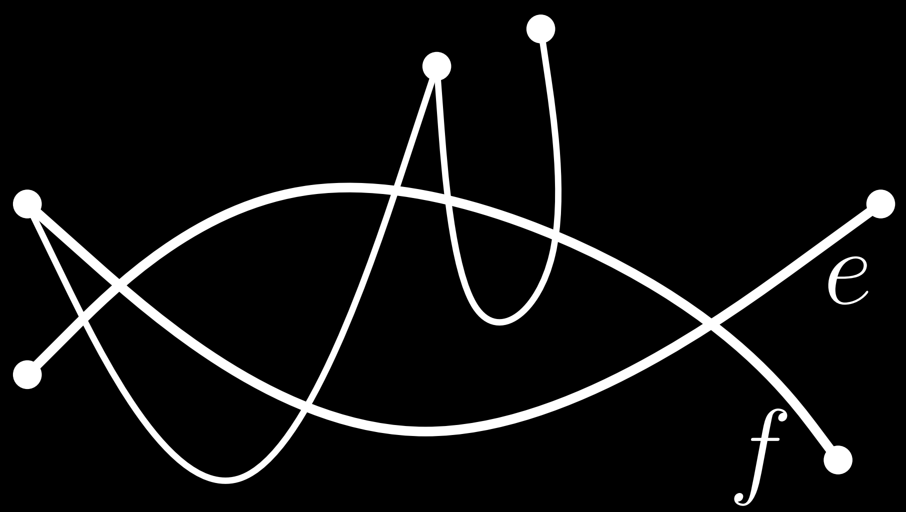
Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

Proof. By Kuratowski theorem it is enough to prove the claim for subdivisions of K_5 and $K_{3,3}$. $\square$

Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

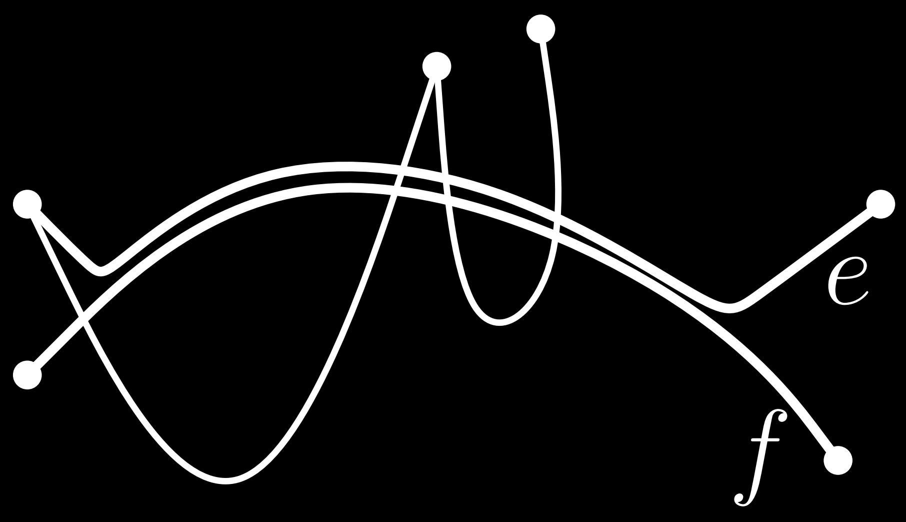
Proof. By Kuratowski theorem it is enough to prove the claim for subdivisions of K_5 and $K_{3,3}$. $\square$



Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

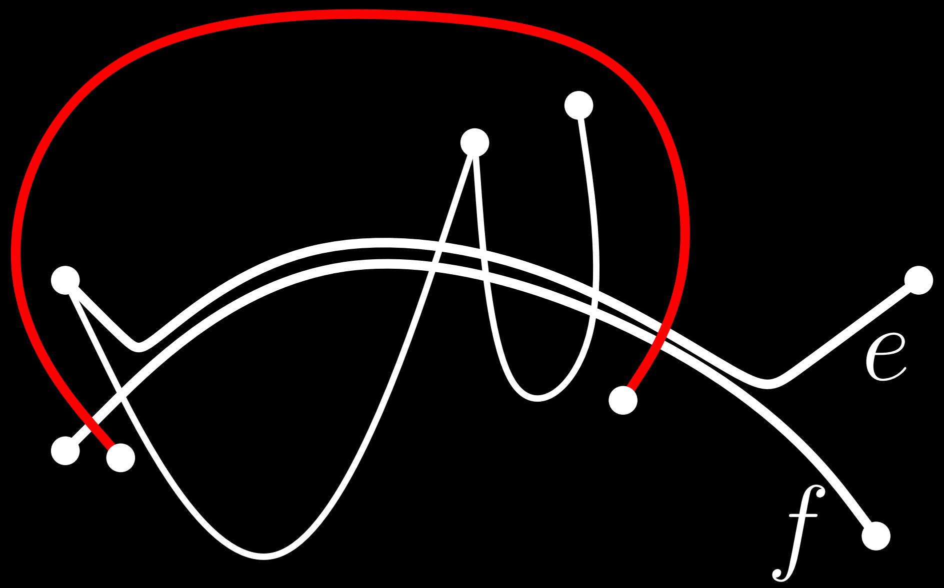
Proof. By Kuratowski theorem it is enough to prove the claim for subdivisions of K_5 and $K_{3,3}$. $\square$



Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

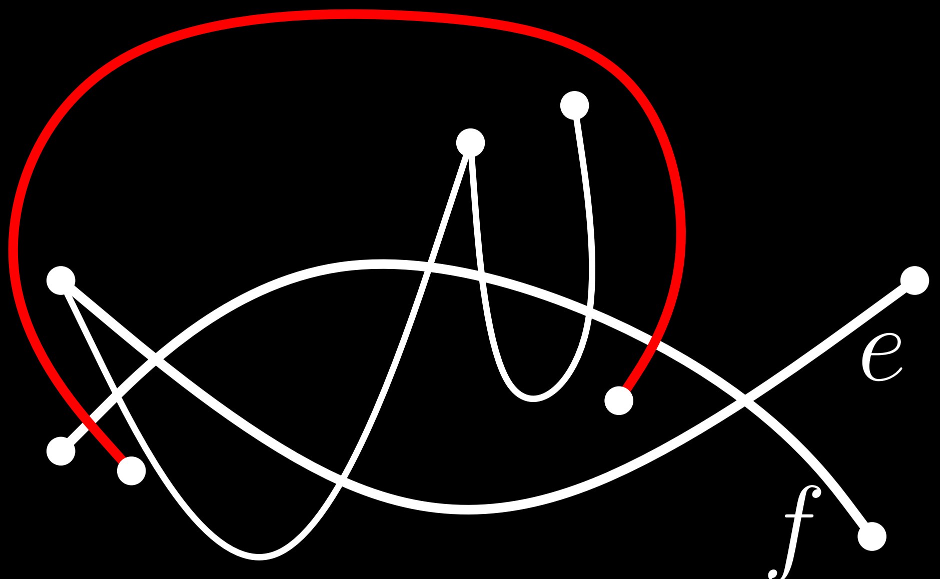
Proof. By Kuratowski theorem it is enough to prove the claim for subdivisions of K_5 and $K_{3,3}$. □



Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

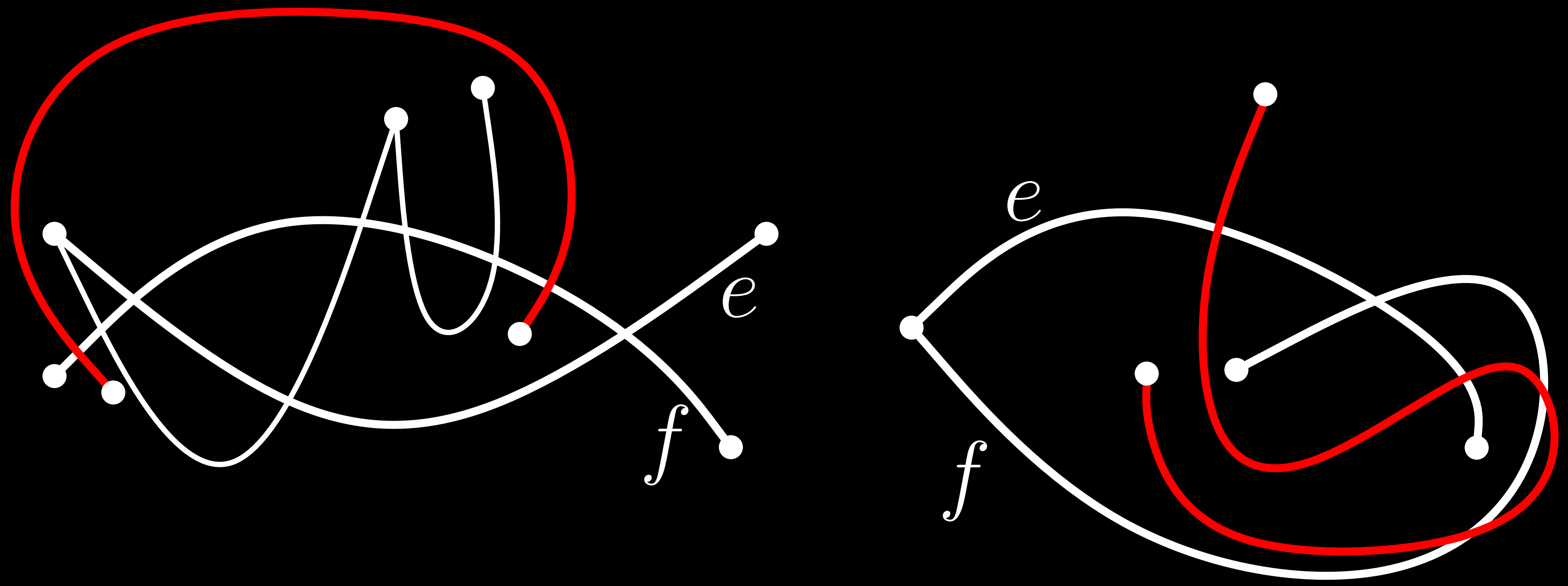
Proof. By Kuratowski theorem it is enough to prove the claim for subdivisions of K_5 and $K_{3,3}$. □



Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

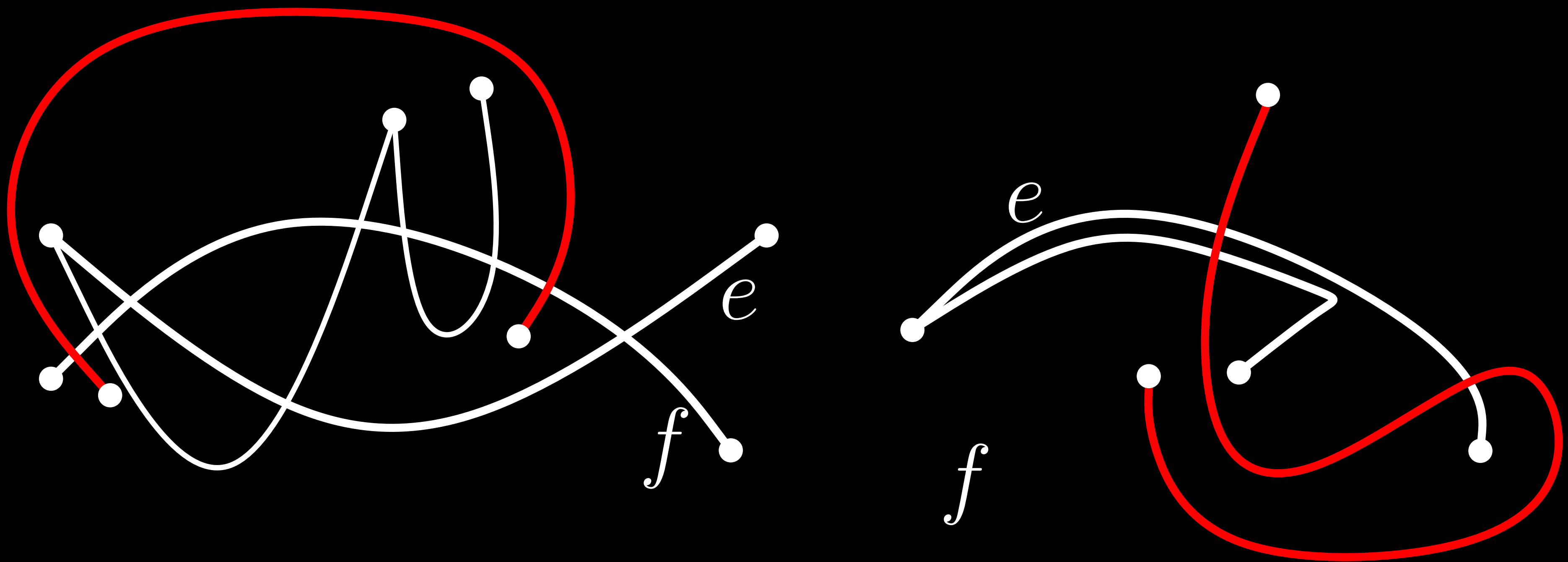
Proof. By Kuratowski theorem it is enough to prove the claim for subdivisions of K_5 and $K_{3,3}$. □



Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

Proof. By Kuratowski theorem it is enough to prove the claim for subdivisions of K_5 and $K_{3,3}$. □



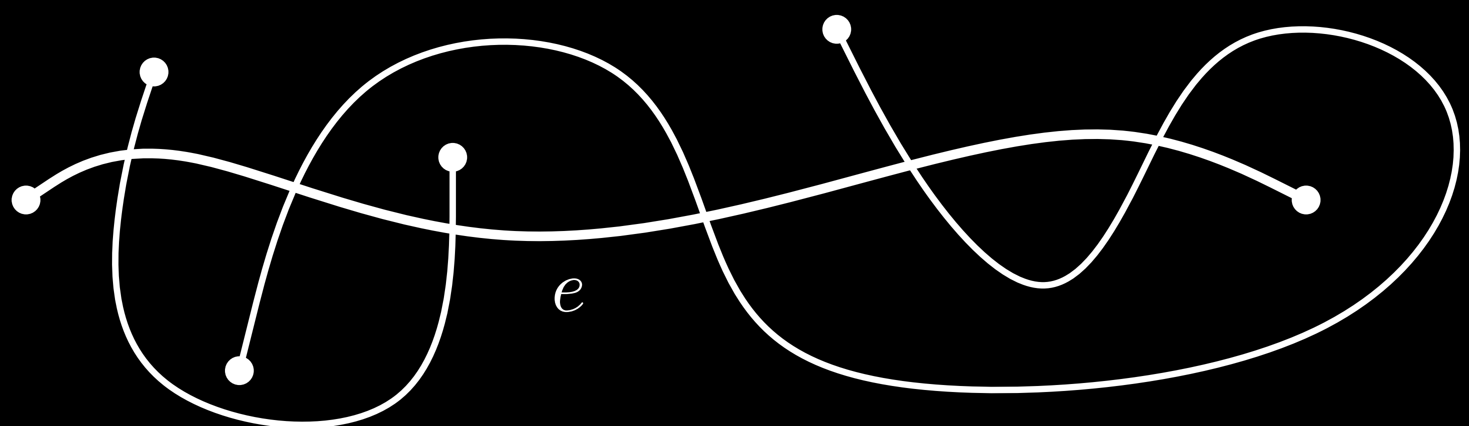
Cleaning even edges

Cleaning even edges

Pach & Tóth (2000), F, Pelsmajer, Schaefer and Štefankovič (2012): Let G denote a graph drawn in the plane. Let $e \in E(G)$ such that e is crossed by every other edge an even number of times. We can redraw G, so that e is crossing free, without introducing a new pair of edges crossing an odd number of times.

Cleaning even edges

Pach & Tóth (2000), F, Pelsmajer, Schaefer and Štefankovič (2012): Let G denote a graph drawn in the plane. Let $e \in E(G)$ such that e is crossed by every other edge an even number of times. We can redraw G, so that e is crossing free, without introducing a new pair of edges crossing an odd number of times.

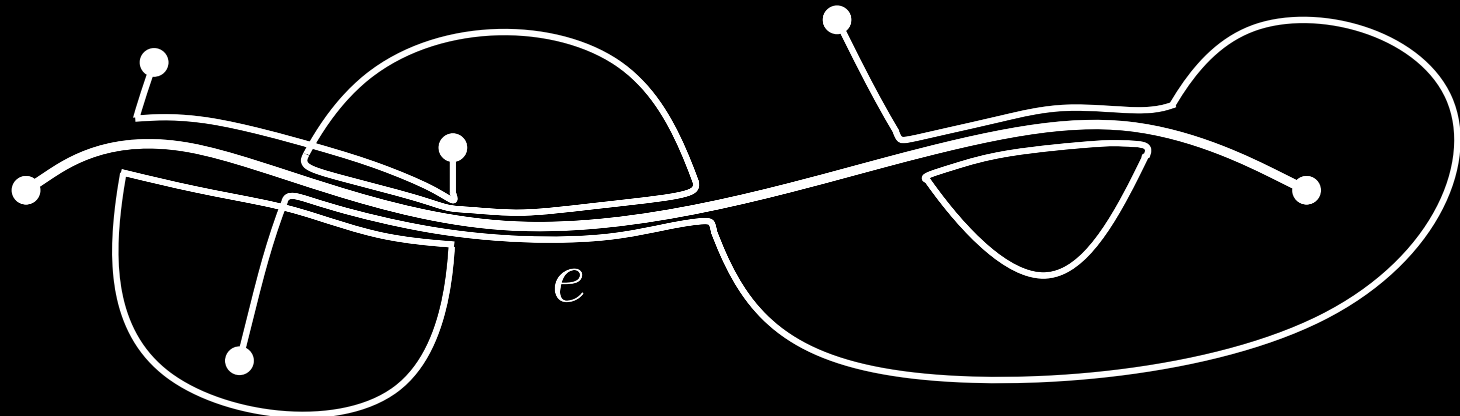


Cleaning even edges

Pach & Tóth (2000), F, Pelsmajer, Schaefer and Štefankovič (2012): Let G denote a graph drawn in the plane. Let $e \in E(G)$ such that e is crossed by every other edge an even number of times. We can redraw G, so that e is crossing free, without introducing a new pair of edges crossing an odd number of times.

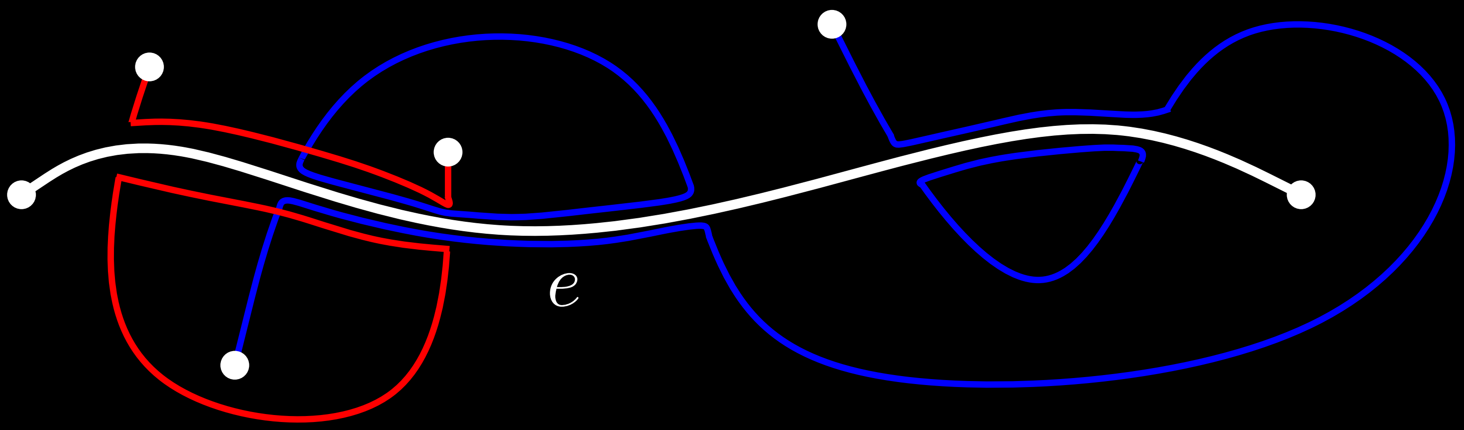
Cleaning even edges

Pach & Tóth (2000), F, Pelsmajer, Schaefer and Štefankovič (2012): Let G denote a graph drawn in the plane. Let $e \in E(G)$ such that e is crossed by every other edge an even number of times. We can redraw G, so that e is crossing free, without introducing a new pair of edges crossing an odd number of times.



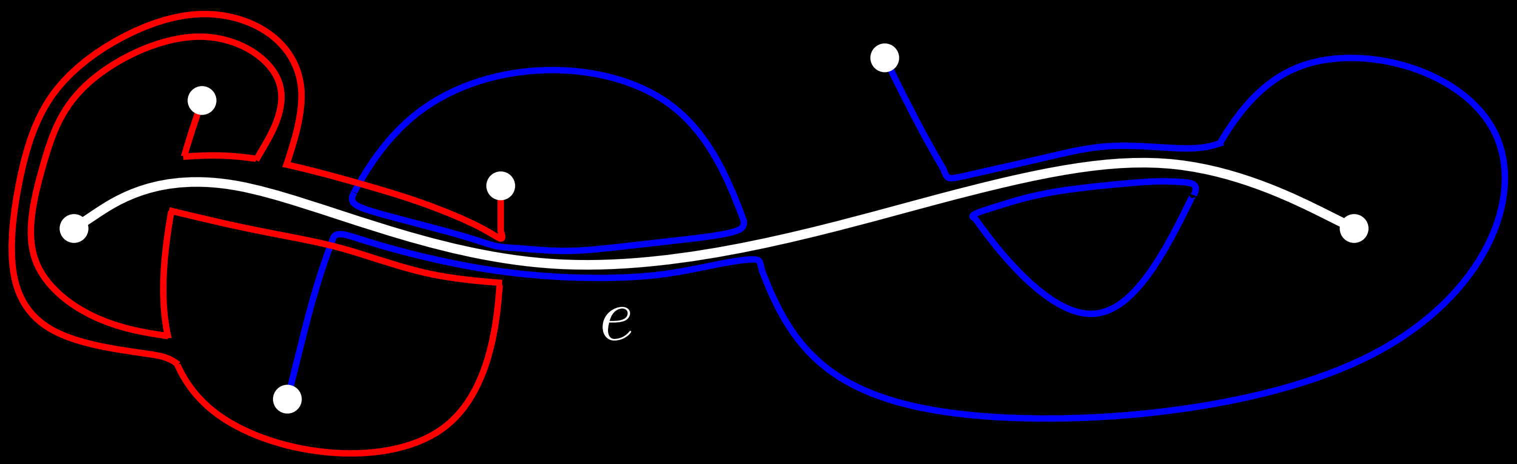
Cleaning even edges

Pach & Tóth (2000), F, Pelsmajer, Schaefer and Štefankovič (2012): Let G denote a graph drawn in the plane. Let $e \in E(G)$ such that e is crossed by every other edge an even number of times. We can redraw G, so that e is crossing free, without introducing a new pair of edges crossing an odd number of times.



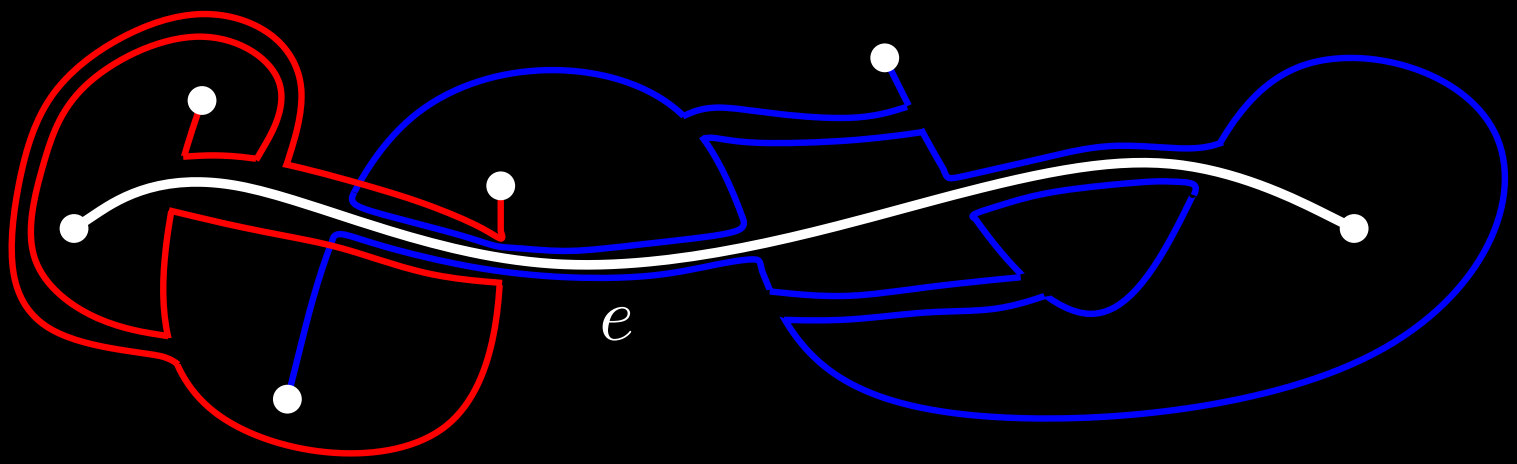
Cleaning even edges

Pach & Tóth (2000), F, Pelsmajer, Schaefer and Štefankovič (2012): Let G denote a graph drawn in the plane. Let $e \in E(G)$ such that e is crossed by every other edge an even number of times. We can redraw G, so that e is crossing free, without introducing a new pair of edges crossing an odd number of times.



Cleaning even edges

Pach & Tóth (2000), F, Pelsmajer, Schaefer and Štefankovič (2012): Let G denote a graph drawn in the plane. Let $e \in E(G)$ such that e is crossed by every other edge an even number of times. We can redraw G, so that e is crossing free, without introducing a new pair of edges crossing an odd number of times.



Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

Cairns & Nikolayevsky (2000): A graph is *embeddable on an* **orientable** *surface* if it can be drawn in the surface such that any two edges cross an **even** number of times.

Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

Cairns & Nikolayevsky (2000)+Pelsmajer, Schaefer, and Štefankovič (2007): A graph is *embeddable on a surface* if it can be drawn in the surface such that any two edges cross an **even** number of times. Moreover, in the embedding the order of the end pieces of the edges at the vertices is the same.

Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

Cairns & Nikolayevsky (2000)+Pelsmajer, Schaefer, and Štefankovič (2007): A graph is *embeddable on a surface* if it can be drawn in the surface such that any two edges cross an **even** number of times. Moreover, in the embedding the order of the end pieces of the edges at the vertices is the same.

Conjecture: A graph is *embeddable on a surface* if it can be drawn in the surface such that any two **non-adjacent** edges cross an **even** number of times.

Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

Cairns & Nikolayevsky (2000)+Pelsmajer, Schaefer, and Štefankovič (2007): A graph is *embeddable on a surface* if it can be drawn in the surface such that any two edges cross an **even** number of times. Moreover, in the embedding the order of the end pieces of the edges at the vertices is the same.

Conjecture: A graph is *embeddable on a surface* if it can be drawn in the surface such that any two **non-adjacent** edges cross an **even** number of times.

Verified for the projective plane by *Pelsmajer et al.* (2009), de Verdiere et al. (2016).

Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

Shapiro (1957), Wu (1970s): Given a graph $G = (V, E)$. We can test in a polynomial time whether G can be drawn so that **no two non-adjacent** edges of G cross an **odd** number of times.

Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

Shapiro (1957), Wu (1970s): Given a graph $G = (V, E)$. We can test in a polynomial time whether G can be drawn so that **no two non-adjacent** edges of G cross an **odd** number of times.

The algorithm reduces the problem to solving a sparse linear system over $\mathbb{Z}/2\mathbb{Z}$ with $O(|V|^2)$ variables and $O(|V|^2)$ equations solvable in $\tilde{O}(|V|^4)$, *Wiedemann (1986)*.

Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

Shapiro (1957), Wu (1970s): Given a graph $G = (V, E)$. We can test in a polynomial time whether G can be drawn so that **no two non-adjacent** edges of G cross an **odd** number of times.

The algorithm reduces the problem to solving a sparse linear system over $\mathbb{Z}/2\mathbb{Z}$ with $O(|V|^2)$ variables and $O(|V|^2)$ equations solvable in $\tilde{O}(|V|^4)$, *Wiedemann (1986)*.

Linear time algorithm was given by *Hopcroft & Tarjan (1970)*

Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

Weak Hanani–Tutte theorem for monotone drawings

Pach & Tóth (2004): If we can draw a graph G in the plane such that
(i) every pair of edges cross evenly; and
(ii) projection $x(.)$ of every edge to x-axis is injective
then we can embed G such that (ii) still holds; $x(v)$ is unchanged for every vertex and the order of the end pieces of the edges at the vertices is unchanged.

Hanani–Tutte theorem

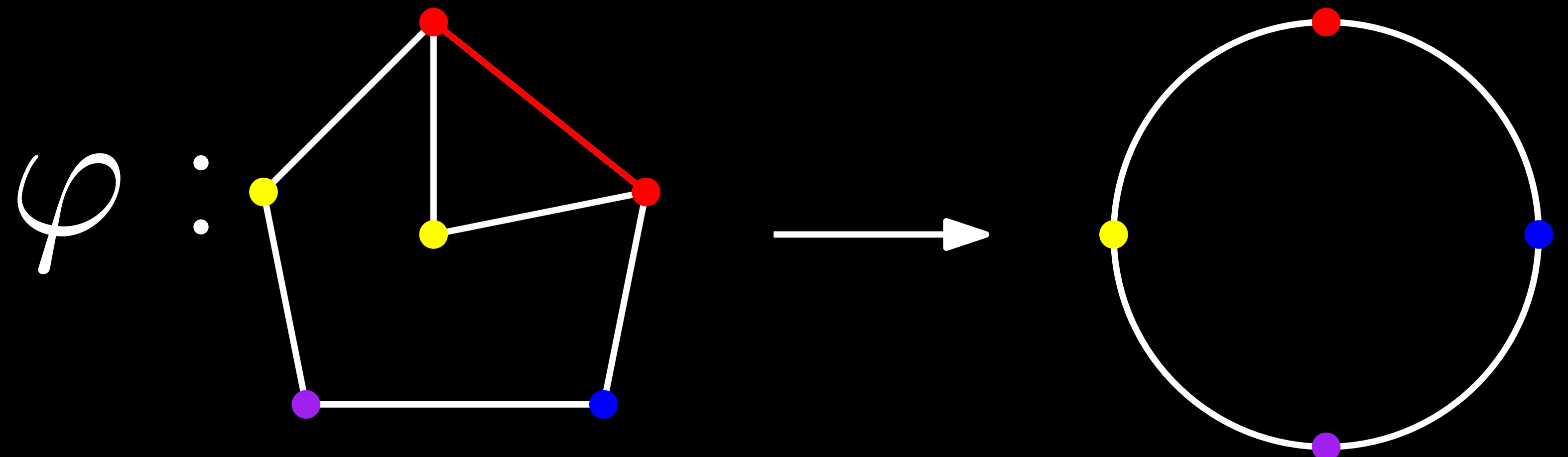
Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

M. Skopenkov (2003): Let T be a graph without vertices of degree > 3. Suppose that T has k vertices. A simplicial map $\varphi : T \to \mathbb{S}^1 \subset \mathbb{R}^2$ is approximable by embeddings if and only if the van Kampen obstruction $v(\varphi) = 0$ and $\varphi^{(k)}$ does not contain standard windings of degree $d \neq \pm 1$, odd.

Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

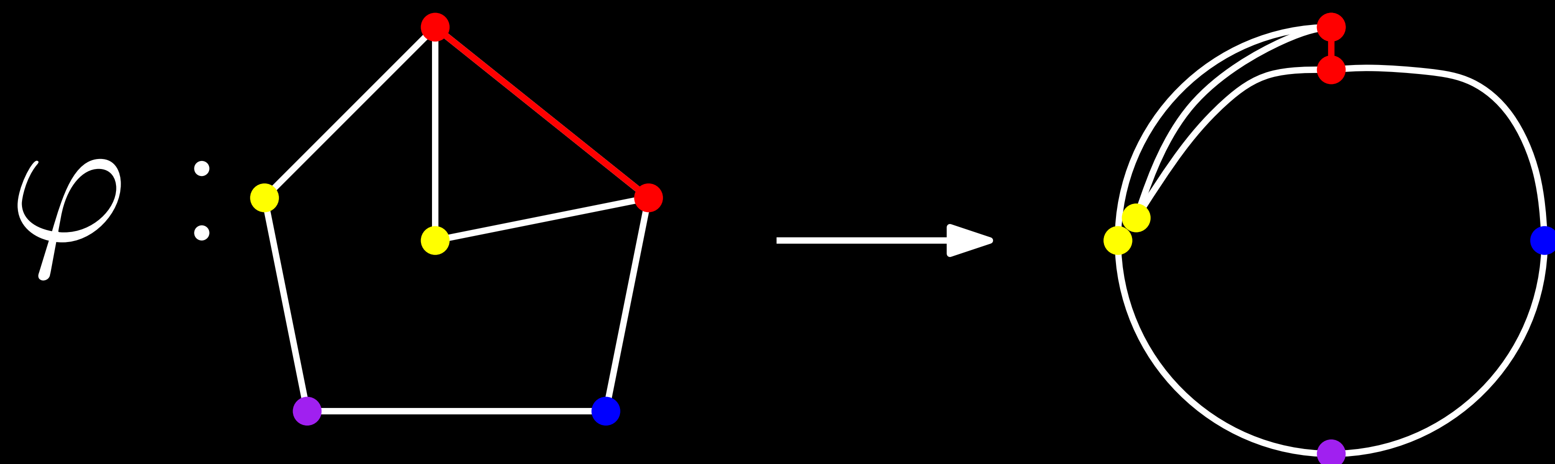
M. Skopenkov (2003): Let T be a graph without vertices of degree > 3. Suppose that T has k vertices. A simplicial map $\varphi : T \to \mathbb{S}^1 \subset \mathbb{R}^2$ is approximable by embeddings if and only if the van Kampen obstruction $v(\varphi) = 0$ and $\varphi^{(k)}$ does not contain standard windings of degree $d \neq \pm 1$, odd.



Hanani–Tutte theorem

Hanani (Chojnacki) (1934), Tutte (1970): A graph is *planar* if it can be drawn in the plane such that any two **non-adjacent** edges cross an **even** number of times.

M. Skopenkov (2003): Let T be a graph without vertices of degree > 3. Suppose that T has k vertices. A simplicial map $\varphi : T \to \mathbb{S}^1 \subset \mathbb{R}^2$ is approximable by embeddings if and only if the van Kampen obstruction $v(\varphi) = 0$ and $\varphi^{(k)}$ does not contain standard windings of degree $d \neq \pm 1$, odd.



Hanani–Tutte theorem for monotone drawings

F., Pelsmajer, Schaefer & Štefankovič (2011): If we can draw a graph G in the plane such that
(i) every pair of **non-adjacent** edges cross evenly; and
(ii) projection $x(.)$ of every edge to x-axis is injective
then we can embed G such that (ii) still holds and $x(v)$ is unchanged for every vertex.

Hanani–Tutte theorem for monotone drawings

F., Pelsmajer, Schaefer & Štefankovič (2011): If we can draw a graph G in the plane such that
(i) every pair of **non-adjacent** edges cross evenly; and
(ii) projection $x(.)$ of every edge to x-axis is injective
then we can embed G such that (ii) still holds and $x(v)$ is unchanged for every vertex.

F., Pelsmajer, Schaefer & Štefankovič (2011): Given a graph $G = (V, E)$ whose vertices are totally ordered we can test in $O(|V|^2)$ time if there exists an embedding of G in which $x(v)$'s, $v \in V$, respect the given order and $x(e)$'s, $e \in E$, are injective

Hanani–Tutte theorem for monotone drawings

F., Pelsmajer, Schaefer & Štefankovič (2011): If we can draw a graph G in the plane such that
(i) every pair of **non-adjacent** edges cross evenly; and
(ii) projection $x(.)$ of every edge to x-axis is injective
then we can embed G such that (ii) still holds and $x(v)$ is unchanged for every vertex.

F., Pelsmajer, Schaefer & Štefankovič (2011): Given a graph $G = (V, E)$ whose vertices are totally ordered we can test in $O(|V|^2)$ time if there exists an embedding of G in which $x(v)$'s, $v \in V$, respect the given order and $x(e)$'s, $e \in E$, are injective

The algorithm reduces the problem to 2-SAT.

Hanani–Tutte theorem for monotone drawings

F., Pelsmajer, Schaefer & Štefankovič (2011): If we can draw a graph G in the plane such that
(i) every pair of **non-adjacent** edges cross evenly; and
(ii) projection $x(.)$ of every edge to x-axis is injective
then we can embed G such that (ii) still holds and $x(v)$ is unchanged for every vertex.

F., Pelsmajer, Schaefer & Štefankovič (2011): Given a graph $G = (V, E)$ whose vertices are totally ordered we can test in $O(|V|^2)$ time if there exists an embedding of G in which $x(v)$'s, $v \in V$, respect the given order and $x(e)$'s, $e \in E$, are injective

The algorithm reduces the problem to 2-SAT.

Linear time algorithm was given by *Jünger & Leipert (2000)*.

Hanani–Tutte theorem for monotone drawings

F., Pelsmajer, Schaefer & Štefankovič (2011): If we can draw a graph G in the plane such that
(i) every pair of **non-adjacent** edges cross evenly; and
(ii) projection $x(.)$ of every edge to x-axis is injective
then we can embed G such that (ii) still holds and $x(v)$ is unchanged for every vertex.

F., Pelsmajer, Schaefer & Štefankovič (2011): Given a graph $G = (V, E)$ whose vertices are totally ordered we can test in $O(|V|^2)$ time if there exists an embedding of G in which $x(v)$'s, $v \in V$, respect the given order and $x(e)$'s, $e \in E$, are injective

The algorithm reduces the problem to 2-SAT.

Chimani et al. (2013) Our algorithm performs well in practice.

Radial Drawings

Radial Drawings

The cylinder $\mathcal{C}$ is $I \times \mathbb{S}^1$, where I is unit interval and $\mathbb{S}^1$ is a unit circle. $I(.)$ is the projection to I.

Given a graph $G = (V, E)$ whose vertices are totally ordered a radial drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $I(v),\ v \in V$, respect the given order; and
(ii) $I(e),\ e \in E$, are injective.

Radial Drawings

The cylinder $\mathcal{C}$ is $I \times \mathbb{S}^1$, where I is unit interval and $\mathbb{S}^1$ is a unit circle. $I(.)$ is the projection to I.

Given a graph $G = (V, E)$ whose vertices are totally ordered a radial drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $I(v),\ v \in V$, respect the given order; and
(ii) $I(e),\ e \in E$, are injective.

Brandenburg et al. (2005) Radial planarity testing can be done in linear time

Radial Drawings

The cylinder $\mathcal{C}$ is $I \times \mathbb{S}^1$, where I is unit interval and $\mathbb{S}^1$ is a unit circle. $I(.)$ is the projection to I.

Given a graph $G = (V, E)$ whose vertices are totally ordered a radial drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $I(v),\ v \in V$, respect the given order; and
(ii) $I(e),\ e \in E$, are injective.

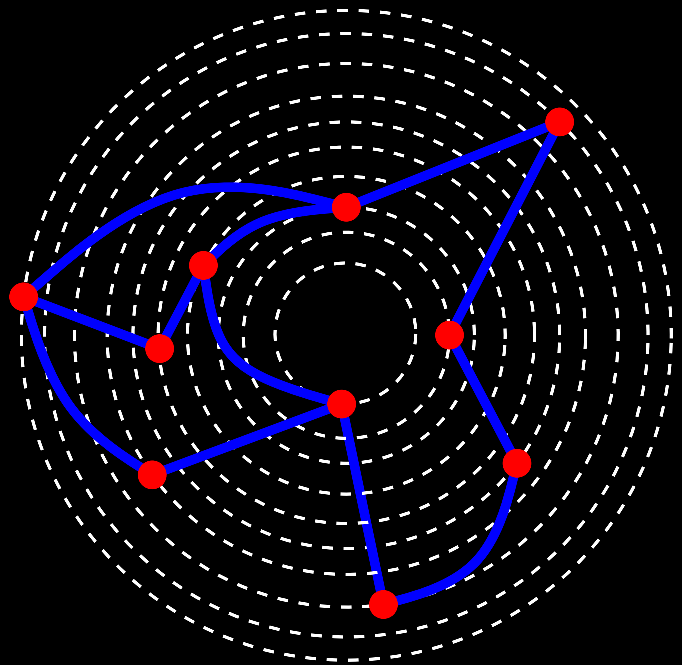
István Orosz

Saratosa

Radial Drawings

The cylinder $\mathcal{C}$ is $I \times \mathbb{S}^1$, where I is unit interval and $\mathbb{S}^1$ is a unit circle. $I(.)$ is the projection to I.

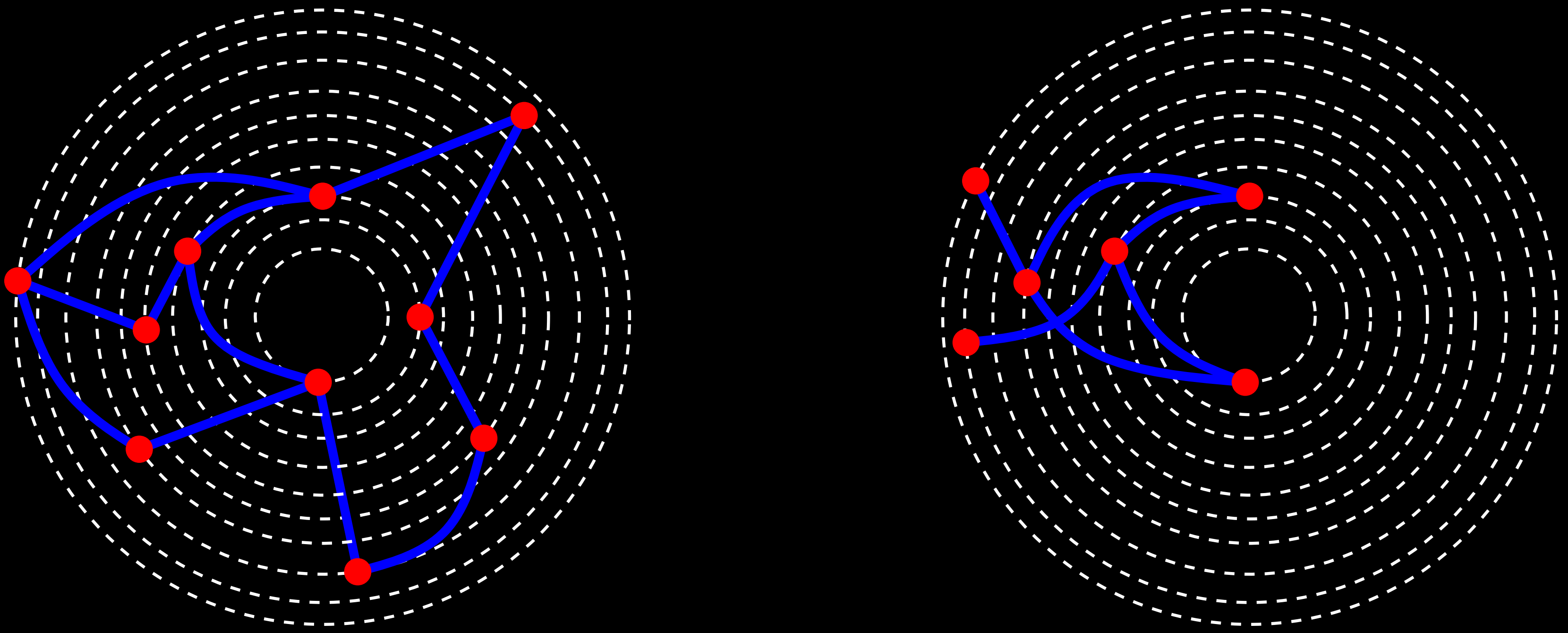
Given a graph $G = (V, E)$ whose vertices are totally ordered a radial drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $I(v),\ v \in V$, respect the given order; and
(ii) $I(e),\ e \in E$, are injective.



Radial Drawings

The cylinder $\mathcal{C}$ is $I \times \mathbb{S}^1$, where I is unit interval and $\mathbb{S}^1$ is a unit circle. $I(.)$ is the projection to I.

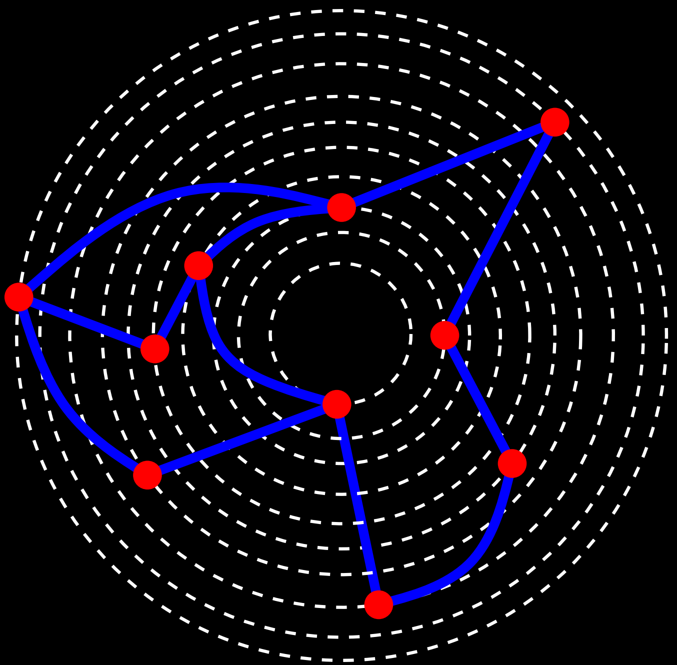
Given a graph $G = (V, E)$ whose vertices are totally ordered a radial drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $I(v),\ v \in V$, respect the given order; and
(ii) $I(e),\ e \in E$, are injective.

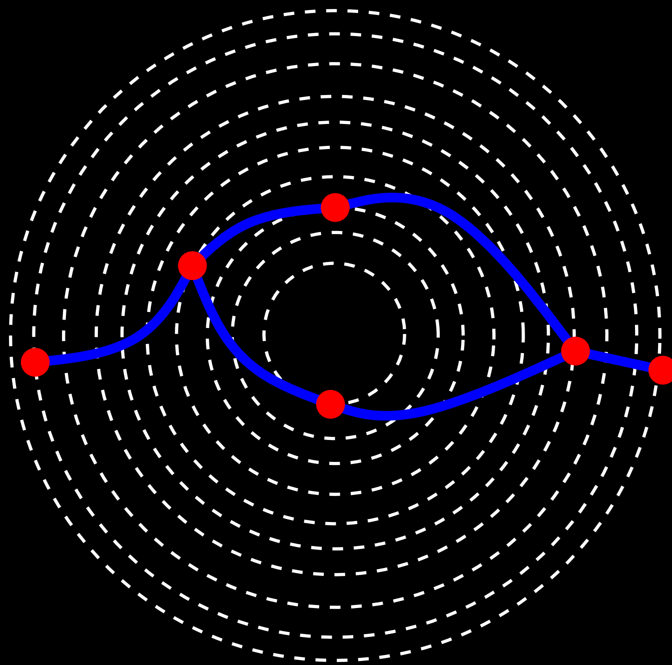


Radial Drawings

The cylinder $\mathcal{C}$ is $I \times \mathbb{S}^1$, where I is unit interval and $\mathbb{S}^1$ is a unit circle. $I(.)$ is the projection to I.

Given a graph $G = (V, E)$ whose vertices are totally ordered a radial drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $I(v),\ v \in V$, respect the given order; and
(ii) $I(e),\ e \in E$, are injective.

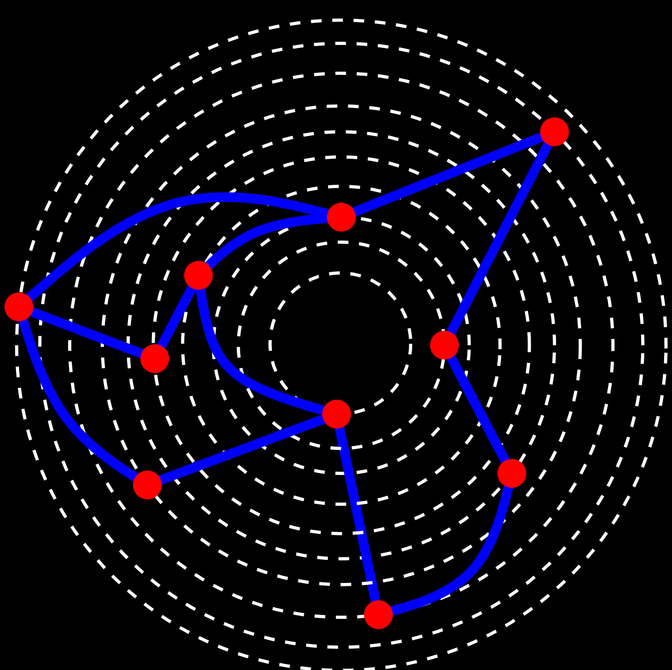


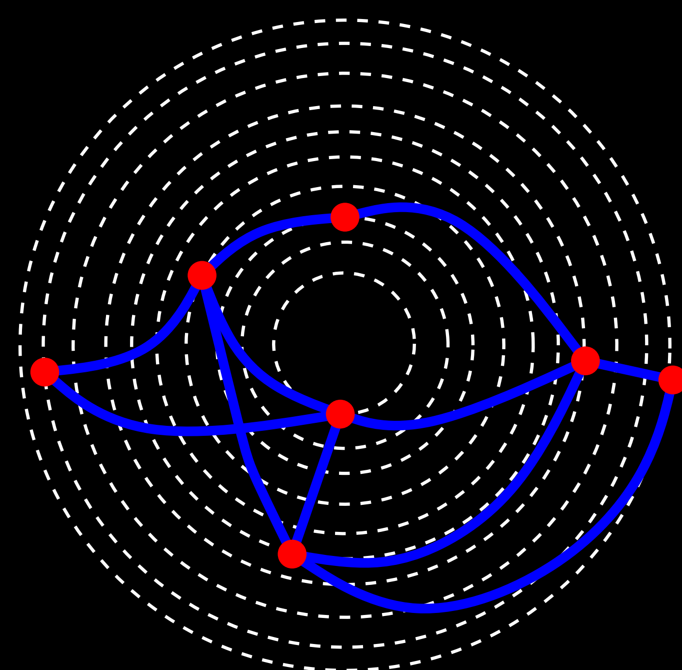


Radial Drawings

The cylinder $\mathcal{C}$ is $I \times \mathbb{S}^1$, where I is unit interval and $\mathbb{S}^1$ is a unit circle. $I(.)$ is the projection to I.

Given a graph $G = (V, E)$ whose vertices are totally ordered a radial drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $I(v),\ v \in V$, respect the given order; and
(ii) $I(e),\ e \in E$, are injective.





Radial Drawings

The cylinder $\mathcal{C}$ is $I \times \mathbb{S}^1$, where I is unit interval and $\mathbb{S}^1$ is a unit circle. $I(.)$ is the projection to I.

Given a graph $G = (V, E)$ whose vertices are totally ordered a radial drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $I(v),\ v \in V$, respect the given order; and
(ii) $I(e),\ e \in E$, are injective.

F., Pelsmajer & Schaefer (2016+): If we can draw a graph G on $\mathcal{C}$ radially such that
(i) every pair of **non-adjacent** edges cross evenly; and
(ii) $I(e)$ is injective for every $e \in E$
then we can embed G radially such that (ii) still holds and $I(v)$ is unchanged for every vertex.

About the proof

About the proof

F., Pelsmajer & Schaefer (2016): If we can draw a graph G on $\mathcal{C}$ radially such that
(i) every pair of **non-adjacent** edges cross evenly; and
(ii) $I(e)$ is injective for every $e \in E$
then we can embed G radially such that (ii) still holds and $I(v)$ is unchanged for every vertex.

About the proof

F., Pelsmajer & Schaefer (2016): If we can draw a graph G on $\mathcal{C}$ radially such that
(i) every pair of **non-adjacent** edges cross evenly; and
(ii) $I(e)$ is injective for every $e \in E$
then we can embed G radially such that (ii) still holds and $I(v)$ is unchanged for every vertex.

Prove the theorem without the word **non-adjacent** and maintaining the order of the end pieces of the edges at the vertices. (*the weak variant*)

About the proof

F., Pelsmajer & Schaefer (2016): If we can draw a graph G on $\mathcal{C}$ radially such that
(i) every pair of **non-adjacent** edges cross evenly; and
(ii) $I(e)$ is injective for every $e \in E$
then we can embed G radially such that (ii) still holds and $I(v)$ is unchanged for every vertex.

Prove the theorem without the word **non-adjacent** and maintaining the order of the end pieces of the edges at the vertices. (*the weak variant*)

Only the first vertex, last vertex and vertices participating in a ≤ 2-cut are locally uncorrectable.

About the proof

F., Pelsmajer & Schaefer (2016): If we can draw a graph G on $\mathcal{C}$ radially such that
(i) every pair of **non-adjacent** edges cross evenly; and
(ii) $I(e)$ is injective for every $e \in E$
then we can embed G radially such that (ii) still holds and $I(v)$ is unchanged for every vertex.

Prove the theorem without the word **non-adjacent** and maintaining the order of the end pieces of the edges at the vertices. (*the weak variant*)

Only the first vertex, last vertex and vertices participating in a ≤ 2-cut are locally uncorrectable.

Reduce ≤ 2-separations; *the weak variant* in the base case.

Limits of Hanani–Tutte

Limits of Hanani–Tutte

The cylinder $\mathcal{C}$ is $I \times \mathbb{S}^1$, where I is unit interval and $\mathbb{S}^1$ is a unit circle. $\mathbb{S}^1(.)$ is the projection to $\mathbb{S}^1$.

Limits of Hanani–Tutte

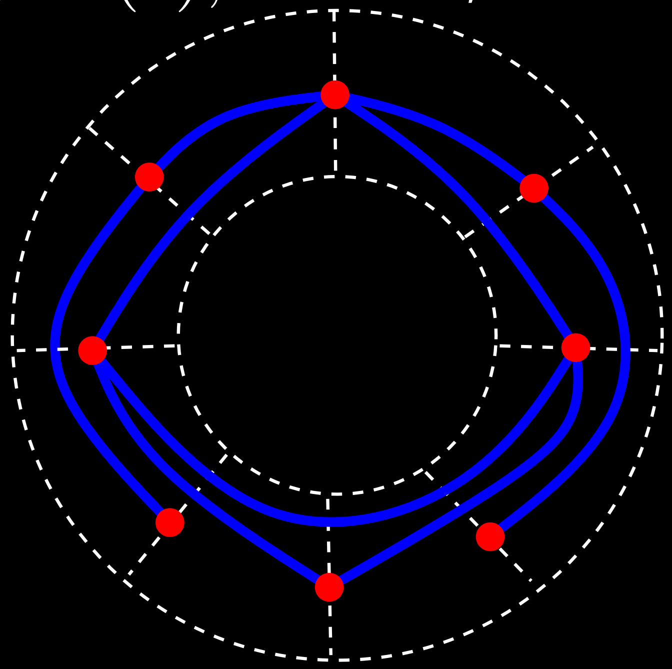
The cylinder $\mathcal{C}$ is $I \times \mathbb{S}^1$, where I is unit interval and $\mathbb{S}^1$ is a unit circle. $\mathbb{S}^1(.)$ is the projection to $\mathbb{S}^1$.

Given a graph $G = (V, \overrightarrow{E})$, whose vertices are cylically ordered, a cyclic drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $\mathbb{S}^1(v),\ v \in V$, respect the given order; and
(ii) $\mathbb{S}^1(e),\ e \in \overrightarrow{E}$, are injective and directed clockwise.

Limits of Hanani–Tutte

The cylinder $\mathcal{C}$ is $I \times \mathbb{S}^1$, where I is unit interval and $\mathbb{S}^1$ is a unit circle. $\mathbb{S}^1(.)$ is the projection to $\mathbb{S}^1$.

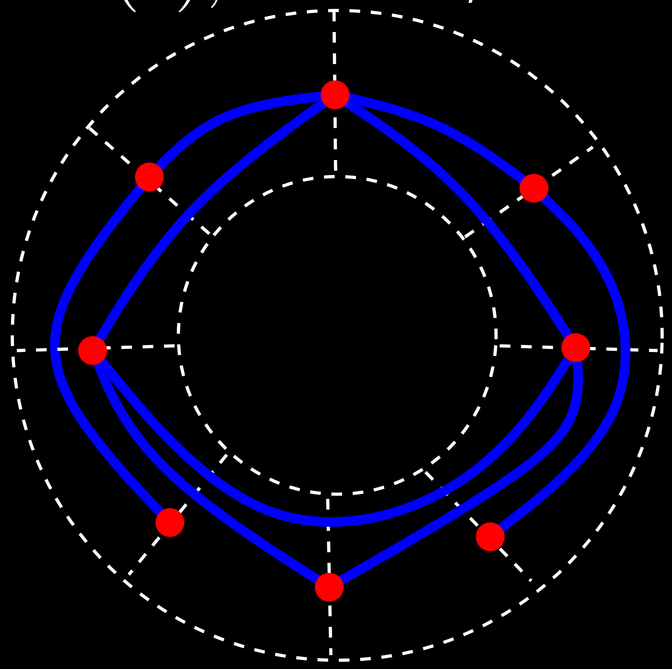
Given a graph $G = (V, \overrightarrow{E})$, whose vertices are cylically ordered, a cyclic drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $\mathbb{S}^1(v)$, $v \in V$, respect the given order; and
(ii) $\mathbb{S}^1(e)$, $e \in \overrightarrow{E}$, are injective and directed clockwise.

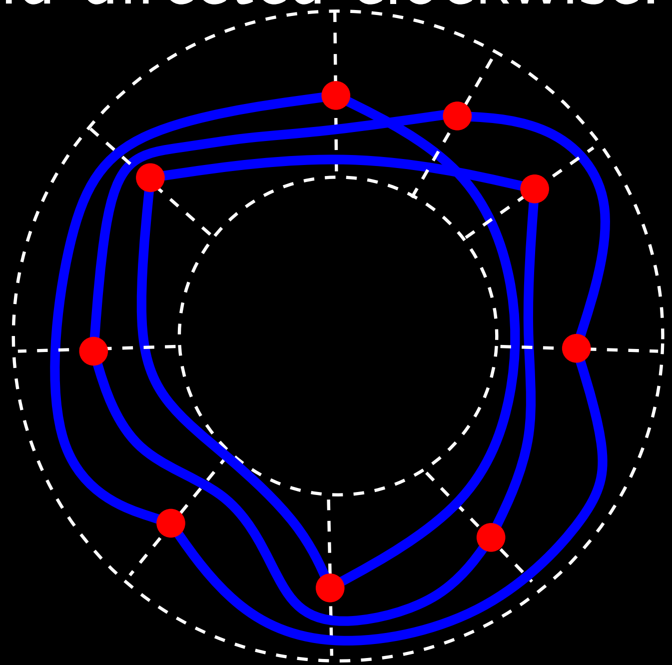


Limits of Hanani–Tutte

The cylinder $\mathcal{C}$ is $I \times \mathbb{S}^1$, where I is unit interval and $\mathbb{S}^1$ is a unit circle. $\mathbb{S}^1(.)$ is the projection to $\mathbb{S}^1$.

Given a graph $G = (V, \overrightarrow{E})$, whose vertices are cylically ordered, a cyclic drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $\mathbb{S}^1(v)$, $v \in V$, respect the given order; and
(ii) $\mathbb{S}^1(e)$, $e \in \overrightarrow{E}$, are injective and directed clockwise.





Limits of Hanani–Tutte

The cylinder $\mathcal{C}$ is $I \times \mathbb{S}^1$, where I is unit interval and $\mathbb{S}^1$ is a unit circle. $\mathbb{S}^1(.)$ is the projection to $\mathbb{S}^1$.

Given a graph $G = (V, \overrightarrow{E})$, whose vertices are cylically ordered, a cyclic drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $\mathbb{S}^1(v)$, $v \in V$, respect the given order; and
(ii) $\mathbb{S}^1(e)$, $e \in \overrightarrow{E}$, are injective and directed clockwise.