EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Karlsruhe Institute of Technology

Beyond Level Planarity

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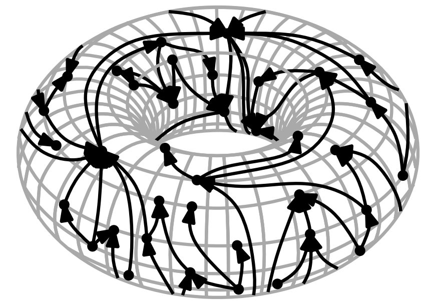
Outline

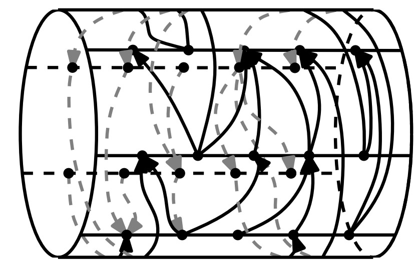
Extensions of Level Planarity

- **Level Embeddings on Surfaces**



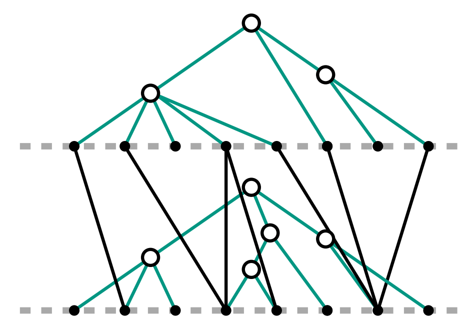
Problems:

- ○ Cyclic and Torus Level Planarity

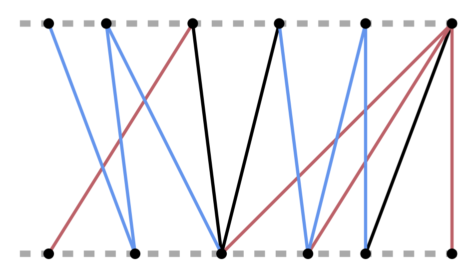


Consecutivity constraints:

- ○ Radial, Cyclic, and Torus $\mathcal{T}$-Level Planarity



- **Simultaneous Level Planarity**

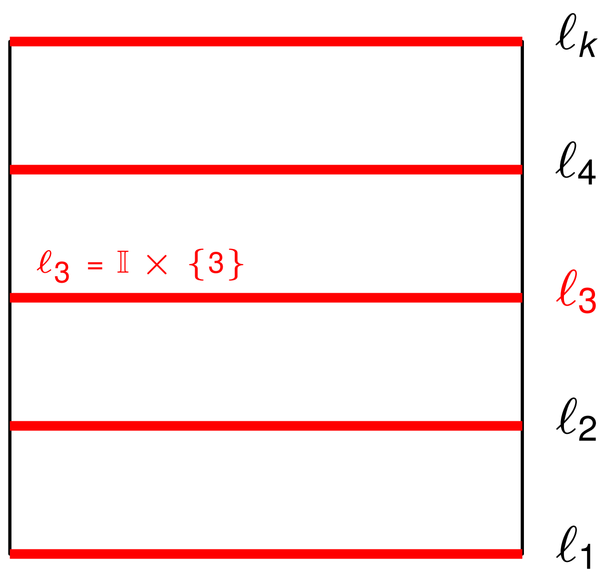


Level Embeddings on Surfaces

(CYCLIC AND TORUS LEVEL PLANARITY)

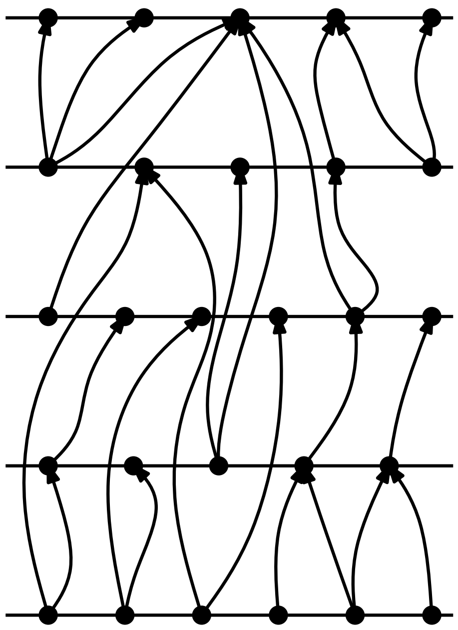
"in order to enlarge the class of level graphs that allow for a level embedding (level drawing with no crossings), the notion of Level Planarity has been extended to surfaces different from the plane"

Levels on the PLANE



$\mathbb{P} = \mathbb{I} \times \mathbb{I}$

Level Planarity

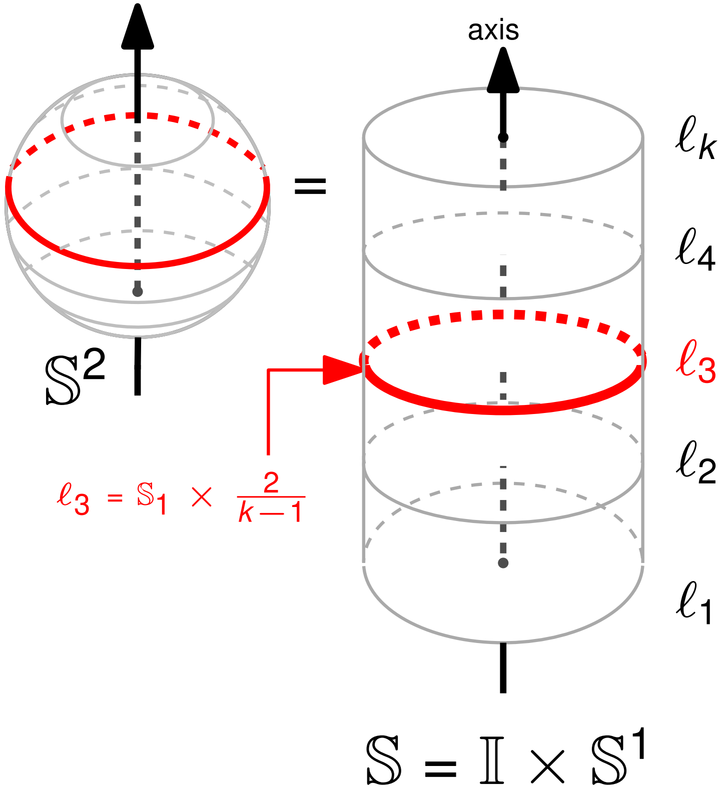


$$(V, E, \gamma), \gamma : V \rightarrow \{1, 2, ..., k\}$$

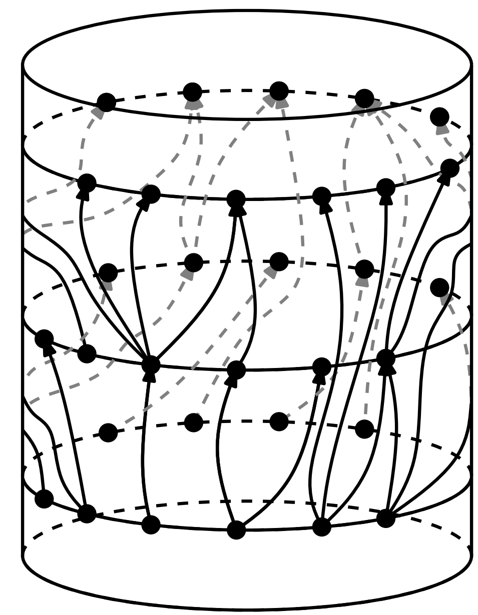
edges (u, v) with $\gamma(u) < \gamma(v)$ are not allowed

- Di Battista, G., Nardelli, E.: **Hierarchies and planarity theory.** *IEEE Trans. Systems, Man, and Cybernetics, 1988.*
- Jünger,M.,Leipert,S.,Mutzel,P.: **Level planarity testing in linear time.** *GD, 1998.*

Levels on the SPHERE/STANDING CYLINDER



Radial Level Planarity



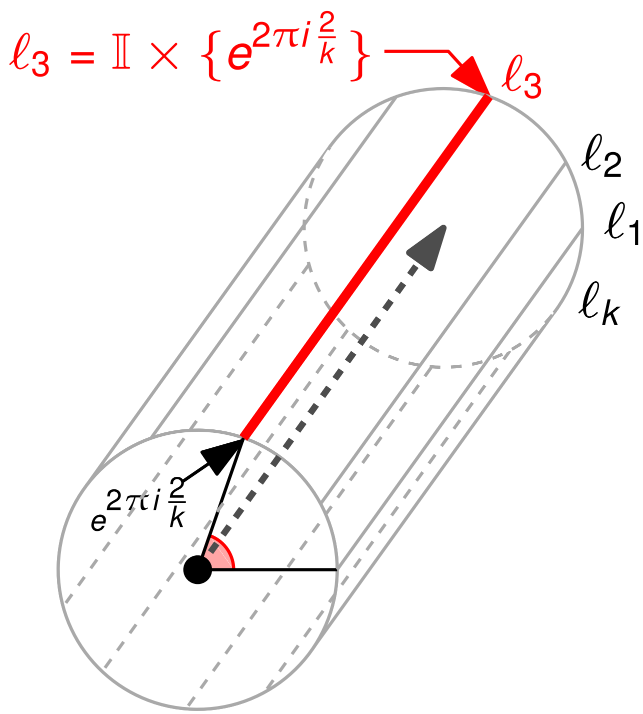
$$(V, E, \gamma), \gamma : V \rightarrow \{1, 2, ..., k\}$$

edges (u, v) with $\gamma(u) < \gamma(v)$ are not allowed

(we can now draw edges that “wrap around” the cylinder axis)

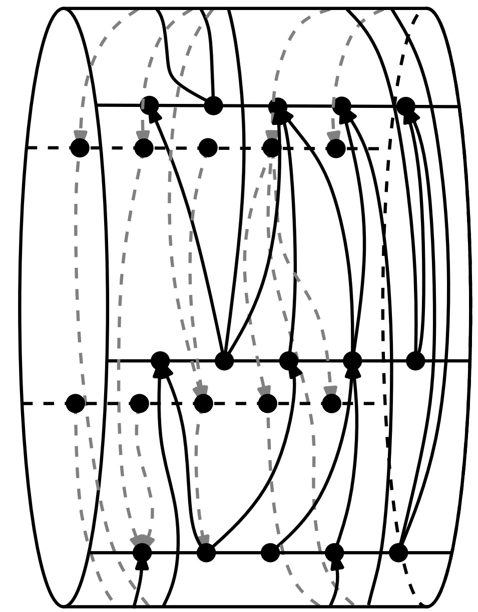
- Bachmaier, C., Brandenburg, F., Forster, M.: **Radial level planarity testing and embedding in linear time**. *JGAA, 2005.*
- Fulek, R., Pelsmajer, M., Schaefer, M.: **Hanani-Tutte for Radial Planarity II**. *GD ’16*

Levels on the ROLLING CYLINDER



$\mathbb{R} = \mathbb{S}^1 \times \mathbb{I}$

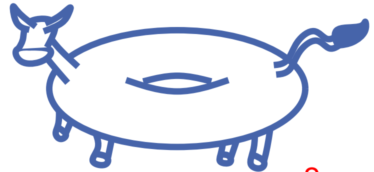
Cyclic Level Planarity

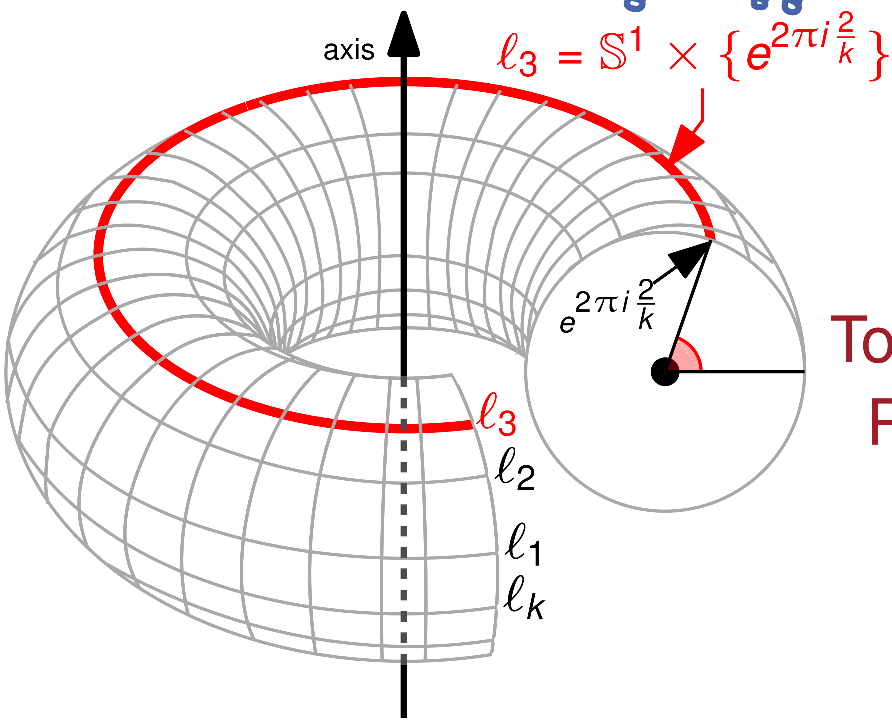


$(V, E, \gamma), \gamma : V \rightarrow \{1, 2, ..., k\}$

edges (u, v) with $\gamma(u) < \gamma(v)$ are allowed

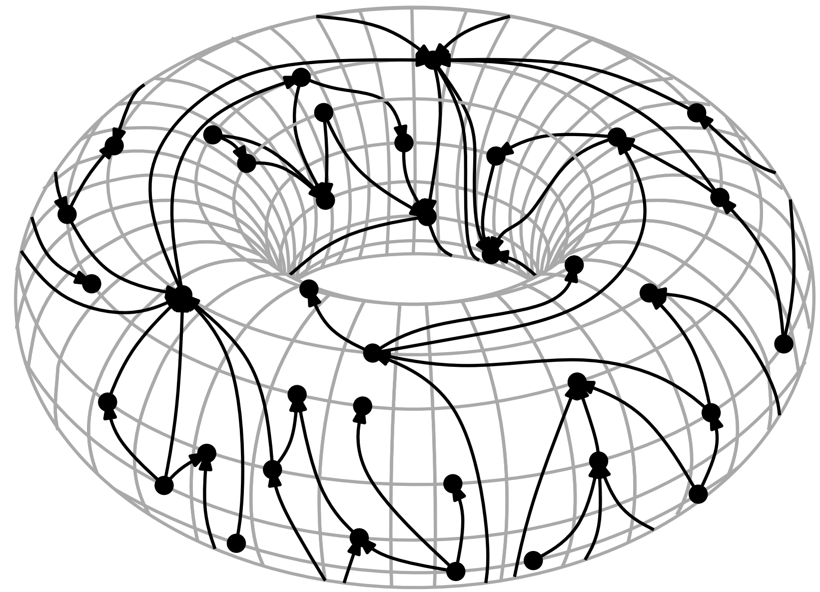
- Bachmaier, C., Brunner, W.: **Linear time planarity testing and embedding of strongly connected cyclic level graphs.** *ESA, 2008.*
- **General instances?** *Bachmaier, Brunner, and König [GD'07] claimed that an $O(|V|^6)$-time algorithm for Cyclic LP can be obtained from the LP testing algorithm by Healy and Kuusik*

Levels on the 



$\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$

Torus Level Planarity



$$(V, E, \gamma), \gamma : V \rightarrow \{1, 2, ..., k\}$$

edges (u, v) with $\gamma(u) < \gamma(v)$ are allowed

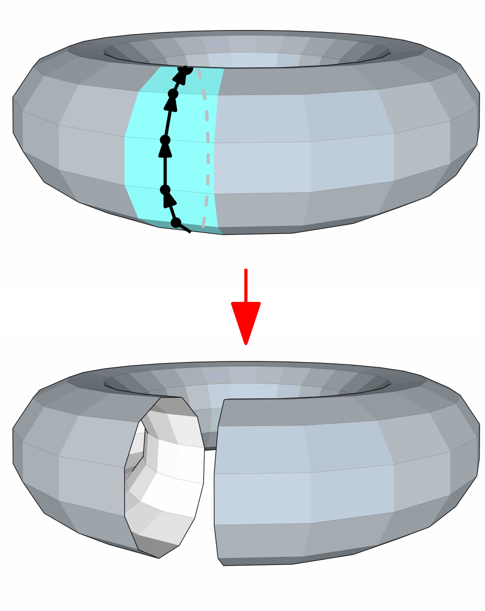
? open problem in:

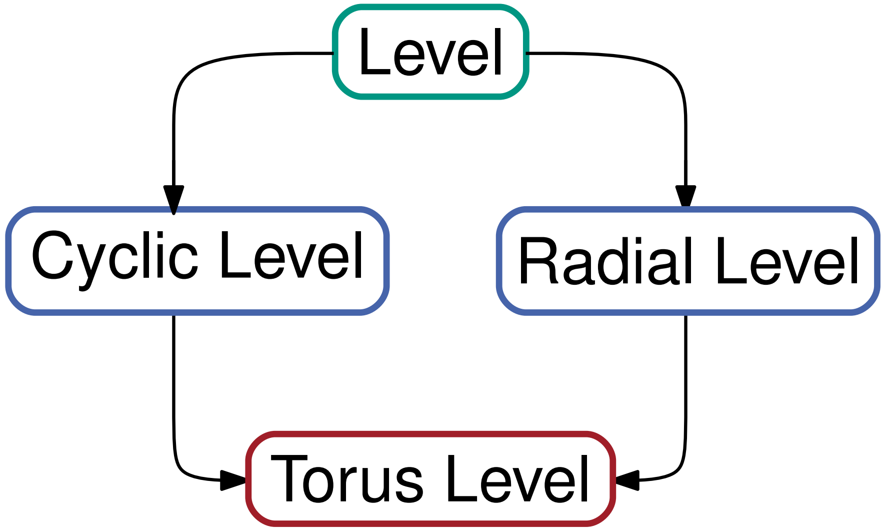
- Bachmaier, C., Brunner, W., König, C.: **Cyclic Level Planarity Testing and Embedding.** *GD '07.*
- Brunner, W.: **Cyclic Level Drawings of Directed Graphs.** *PhD thesis, 2010.*
- Hammersen, K.: **A Characterization of Radial Graphs.** *Deutschen Nationalbibliothek, 2013.*

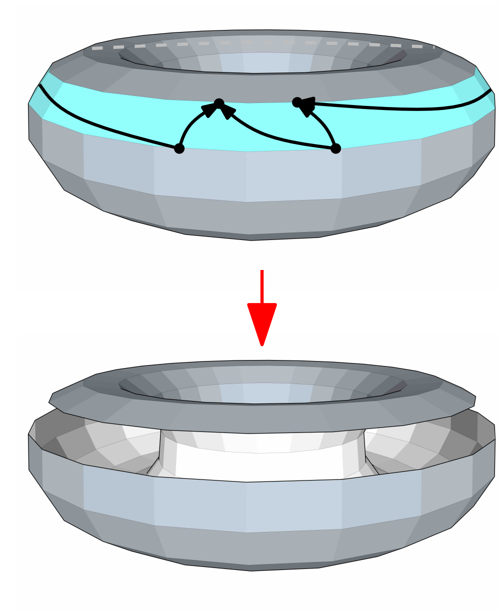
Level Planarity Variants

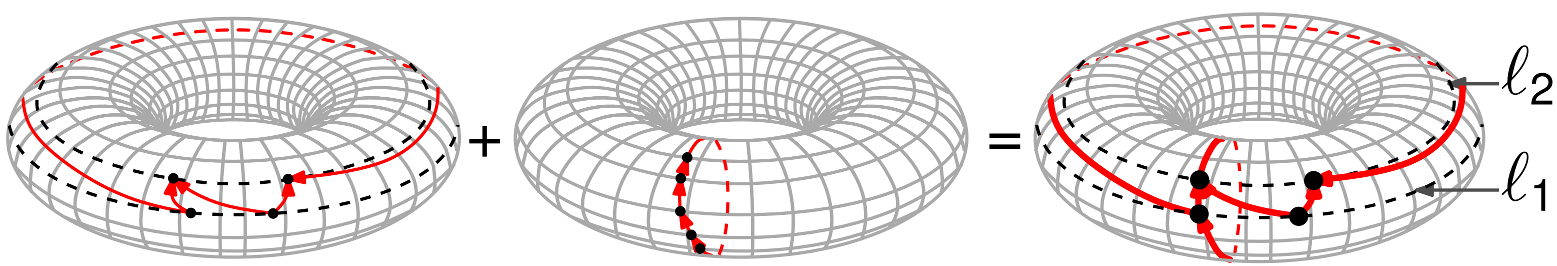
Lemma

CYCLIC AND RADIAL LEVEL PLANARITY $\leq_L$ TORUS LEVEL PLANARITY









A (planar) level graph that is **neither cyclic nor radial level planar**, yet it is **torus level planar**

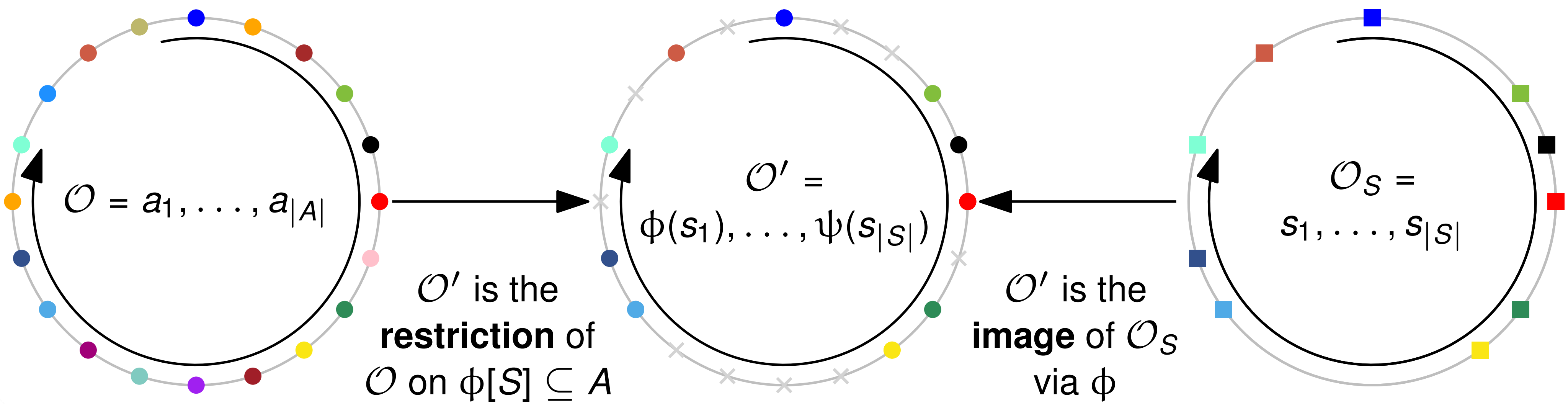
Simultaneous PQ-Ordering: basic concepts 1/2

Lemma

Proper Torus Level Planarity $\leq_L$ Simultaneous PQ-Ordering

- **Orders and Suborders**
  - finite sets A (● ● ● ● ⋯) and S (■ ■ ■ ■ ⋯)
  - **injective map** $\phi : S \to A$
  - order $\mathcal{O}$ on A
  - order $\mathcal{O}_S$ on S

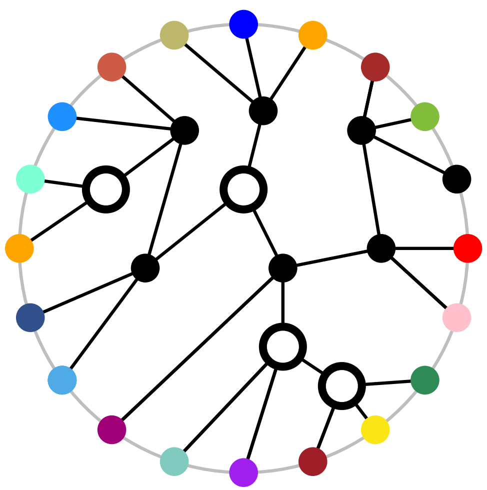
order $\mathcal{O}$ extends order $\mathcal{O}_S$ *when:*



Simultaneous PQ-Ordering: basic concepts 2/2

- **PQ-representable orders**: **(circular) PQ-trees** represent (circular) orders of their leaves with consecutivity constraints
  - **two versions:** `rooted` *[Booth&Lueker, '76]*; `unrooted:` *[Hsu&McConnell, '01]*
  - **two types** of internal nodes

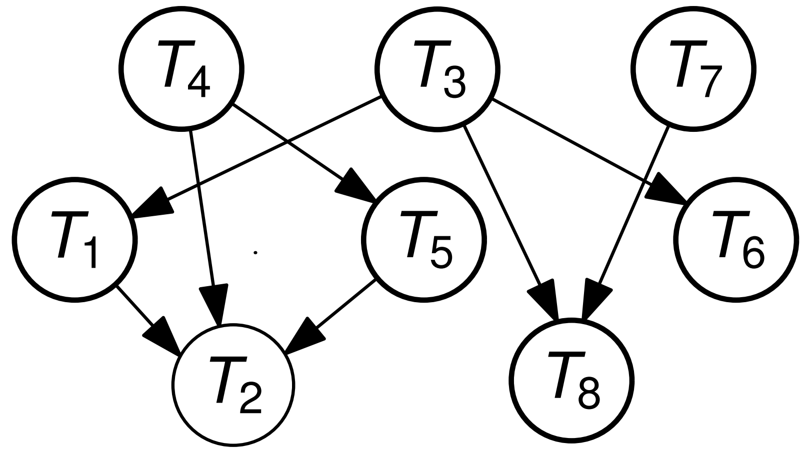
PQ-tree T
with $\mathcal{L}eaves(T) = A$



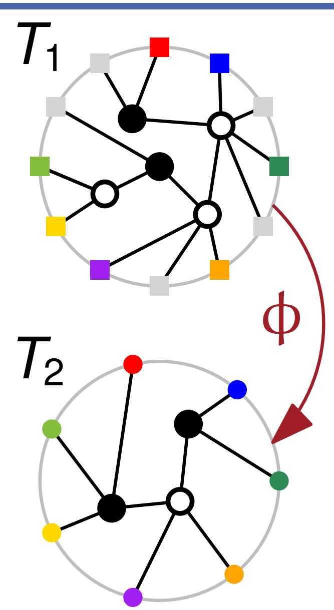
● **P-nodes**: permutations ○ **Q-nodes**: flips

Simultaneous PQ-Ordering: problem definition

- **input:** a DAG = (N, Z)
  - each node $T_i \in N$ is a PQ-tree
  - each arc $\overrightarrow{T_i T_j} \in Z$ is equipped with an **injective map** $\phi : \mathcal{L}eaves(T_j) \to \mathcal{L}eaves(T_i)$



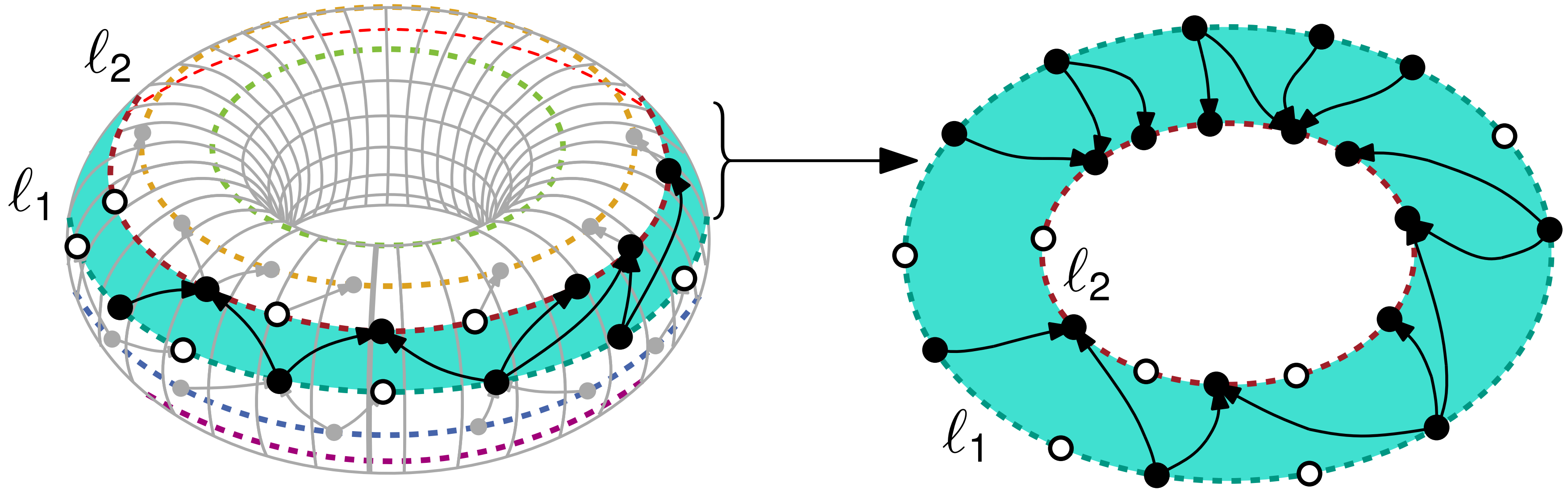
arc
(T_1, T_2, ϕ)
source target



arc (T_1, T_2, ϕ) is **satisfied** by orders $\mathcal{O}_1$ of T_1 and $\mathcal{O}_2$ of T_2 if $\mathcal{O}_1$ extends $\mathcal{O}_2$

- **question:** do there exist orders $\mathcal{O}_i$ for each PQ-tree $T_i \in N$ that simultaneously satisfy all the arcs in Z?

From k levels to 2 levels



Torus Level Embedding of
$G = (\bigcup_{i=1}^{k} V_i, E, \gamma)$

Radial Level Embedding of
$G_{1,2} = (V_i \cup V_{i+1}, (V_i \times V_{i+1}) \cap E, \gamma)$

Observation:

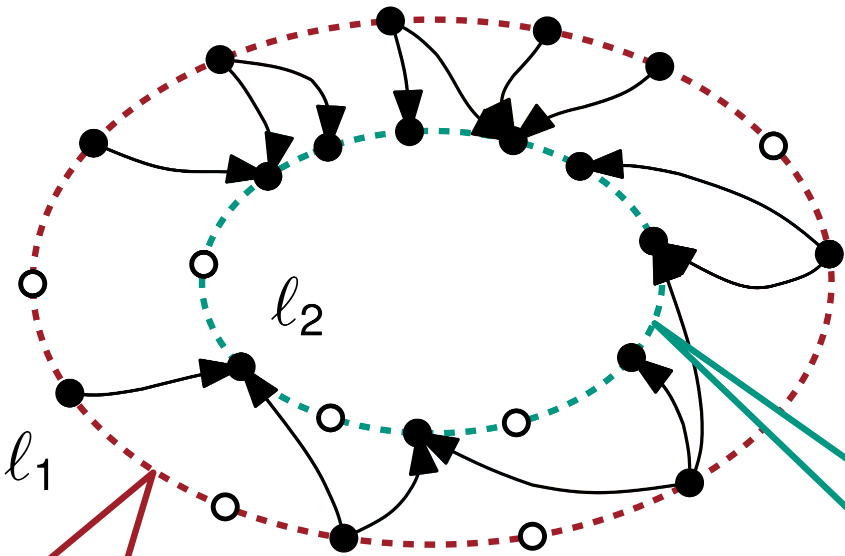
A proper level graph $G = (\bigcup_{i=1}^{k} V_i, E, \gamma)$ has a **torus level embedding** with orders $\mathcal{O}_1, \dots, \mathcal{O}_k$ on $V_1, \dots, V_k$ along $\ell_1, \dots, \ell_k$

if and only if

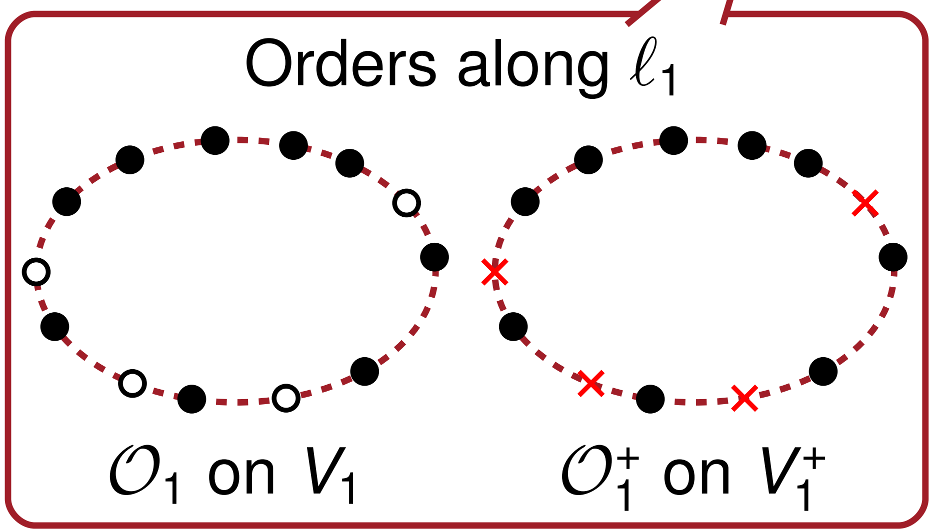
$\exists$ **radial level embedding** of level graphs $(V_i \cup V_{i+1}, (V_i \times V_{i+1}) \cap E, \gamma)$ on two levels with orders $\mathcal{O}_1$ $(\mathcal{O}_{i+1})$ on V_i (V_{i+1}) along ℓ_i (ℓ_{i+1})

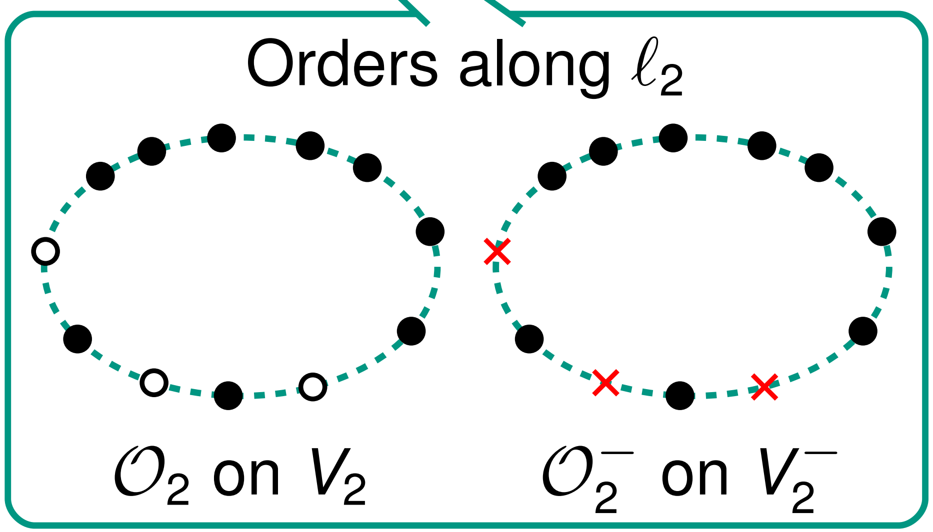
Orders in Radial Level Embeddings: vertex ordering

Level graph $\mathbf{G}_{1,2} = (\mathbf{V}_1 \cup \mathbf{V}_2, \mathbf{E}_{1,2} = \mathbf{E} \cap \mathbf{V}_1 \times \mathbf{V}_2, \gamma)$ between ℓ_1 and ℓ_2



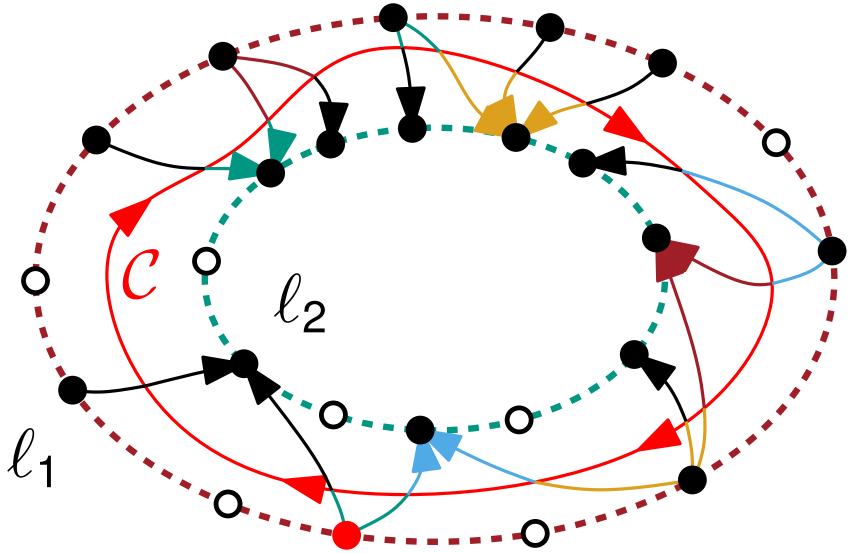
Radial Level Embedding Γ





Orders in Radial Level Embeddings: edge ordering

Level graph $\mathbf{G}_{1,2} = (\mathbf{V}_1 \cup \mathbf{V}_2, \mathbf{E}_{1,2} = \mathbf{E} \cap \mathbf{V}_1 \times \mathbf{V}_2, \gamma)$ between ℓ_1 and ℓ_2



Radial Level Embedding Γ

edge ordering on $E_{1,2}$ ***in*** Γ circular order in which the edges intersect curve $\mathcal{C}$

is ↓

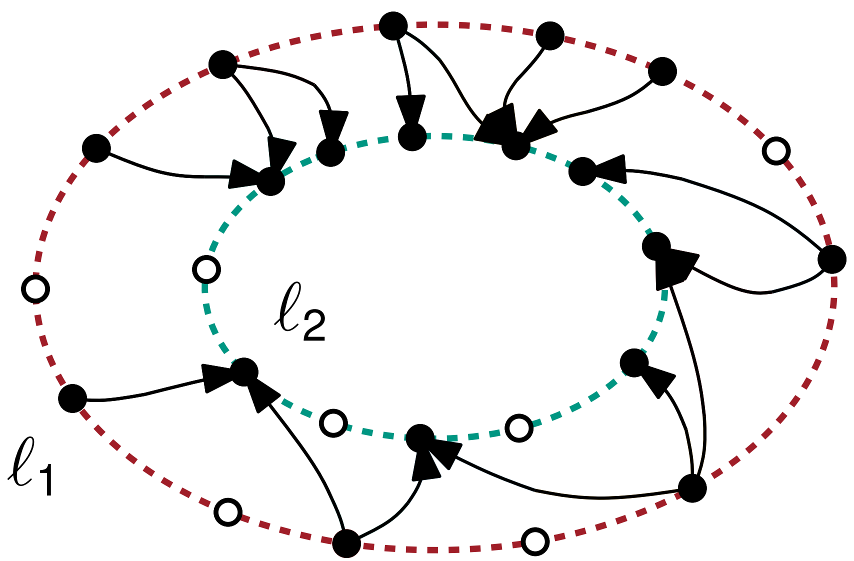
vertex-consecutive order circular order $\mathcal{O}$ on $E_{1,2}$ s.t. $\forall v \in V_1 \cup V_2$ the edges incident to v are **consecutive** in $\mathcal{O}$

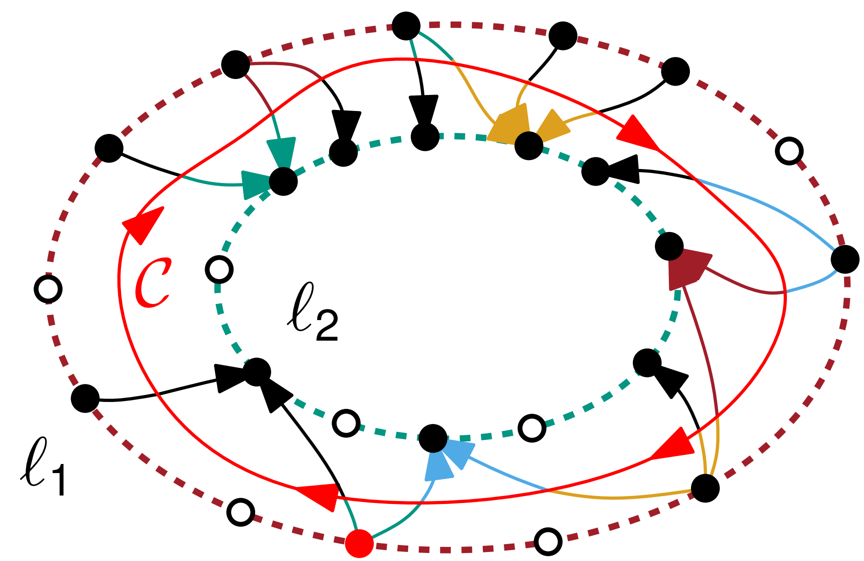
Observation:

vertex-consecutive orders (and hence **edge orders**) are **PQ-representable**

Radial Level Planarity on 2 Levels

Level graph $\mathbf{G}_{1,2} = (\mathbf{V}_1 \cup \mathbf{V}_2, \mathbf{E}_{1,2} = \mathbf{E} \cap \mathbf{V}_1 \times \mathbf{V}_2, \gamma)$ between ℓ_1 and ℓ_2





$\mathcal{O}_i$:= circular order on V_i

$\mathcal{O}$:= circular order on E

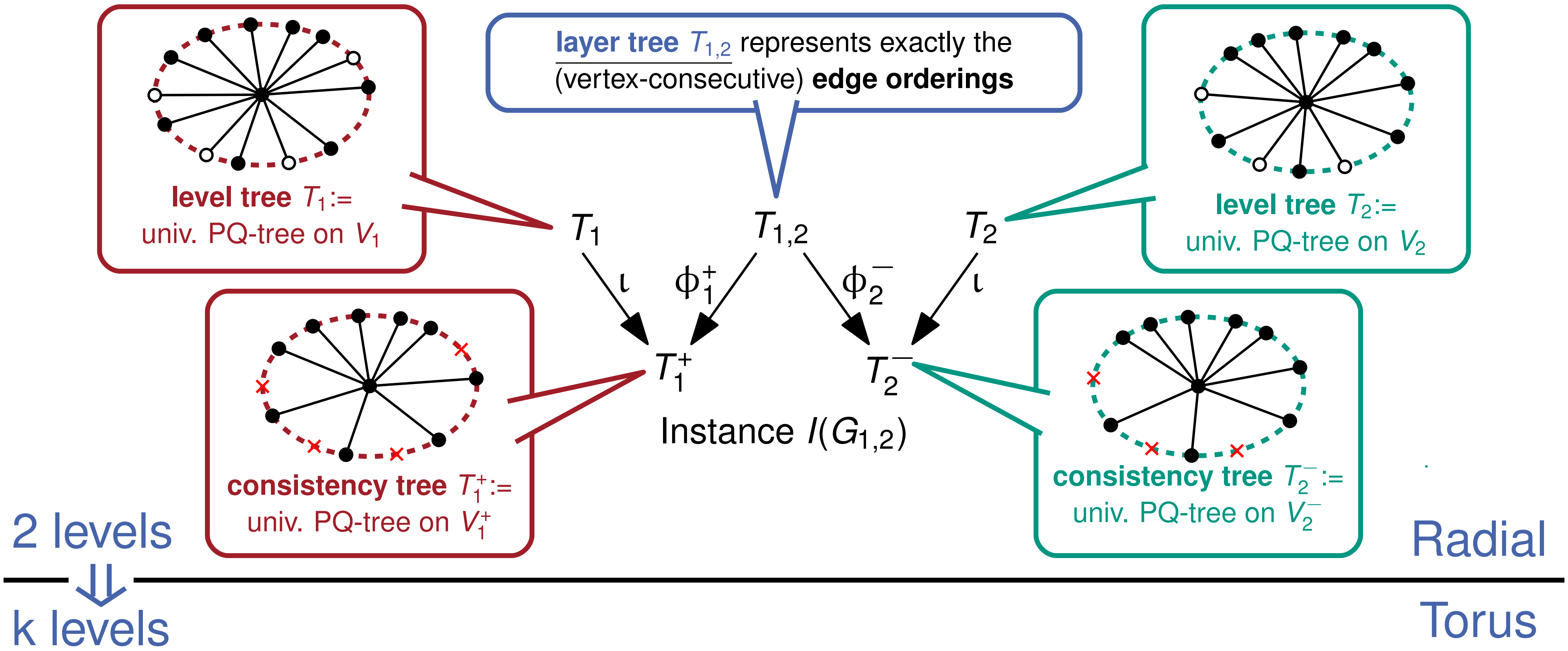
Lemma

$\exists$ **RLE** of $G_{1,2}$ with **edge ordering** $\mathcal{O}$ in which $\mathcal{O}_1$ **and** $\mathcal{O}_2$ **are the orders on** V_1 **and** V_2

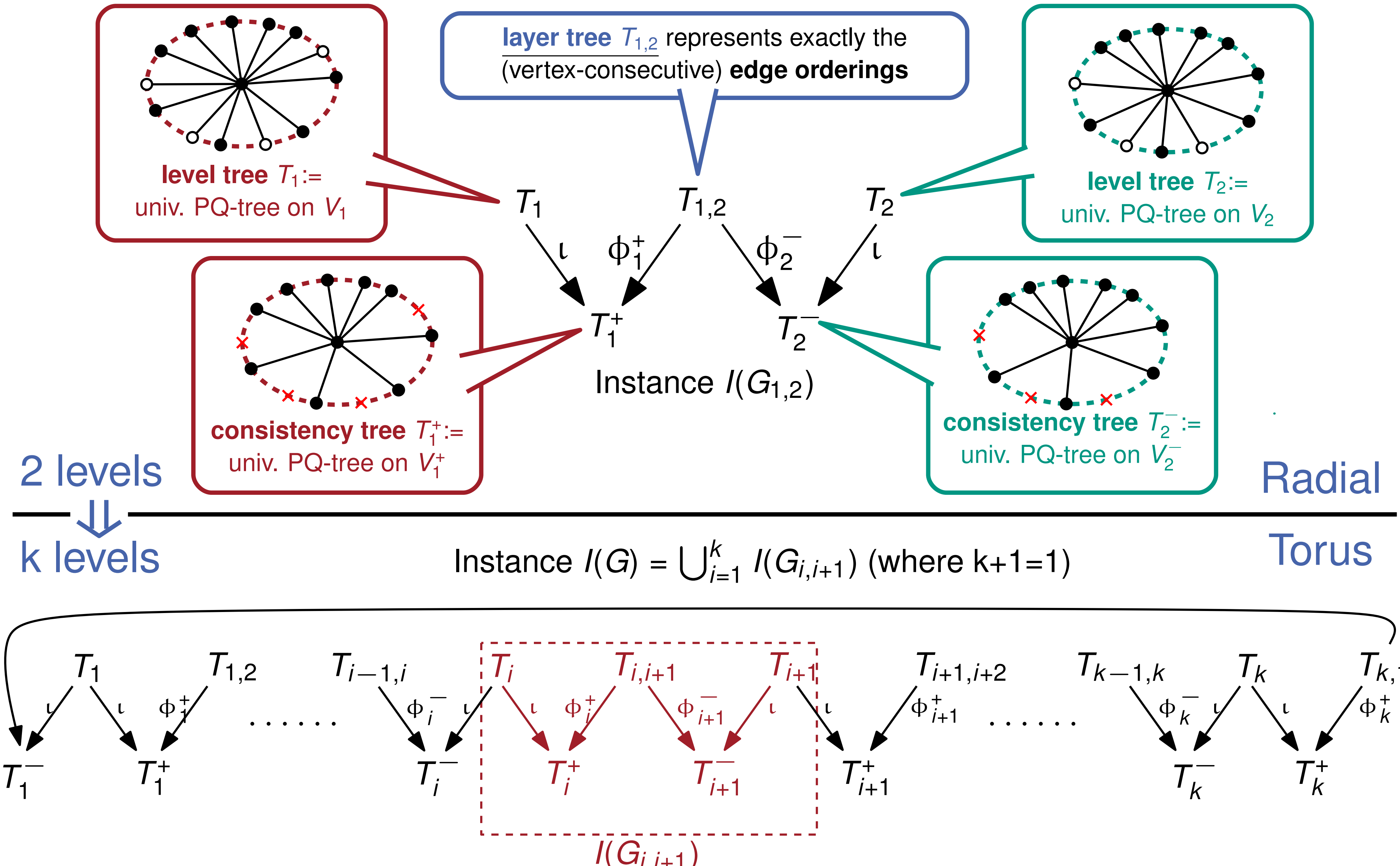
if and only if

- order $\mathcal{O}$ is **vertex-consecutive**
- orders $\mathcal{O}_1$ and $\mathcal{O}_2$ **extend** the orders $\mathcal{O}_1^+$ and $\mathcal{O}_2^-$ on V_1^+ and V_2^- induced by $\mathcal{O}$

Simultaneous PQ-Ordering: from 2 to k levels

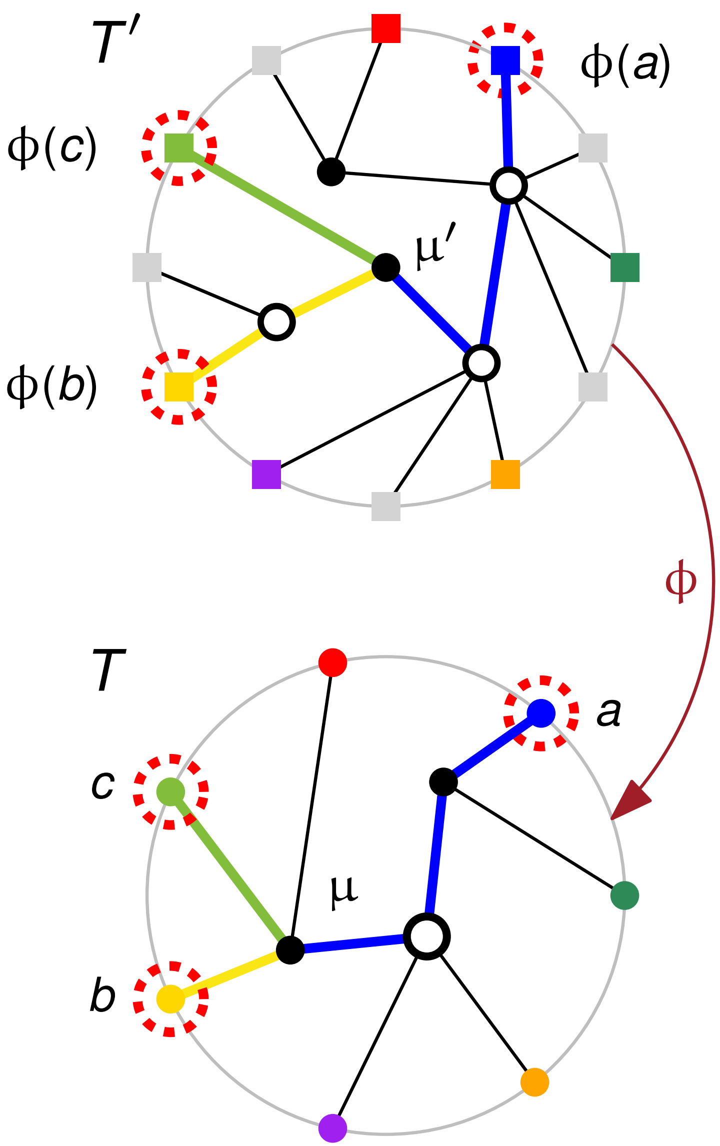


Simultaneous PQ-Ordering: from 2 to k levels



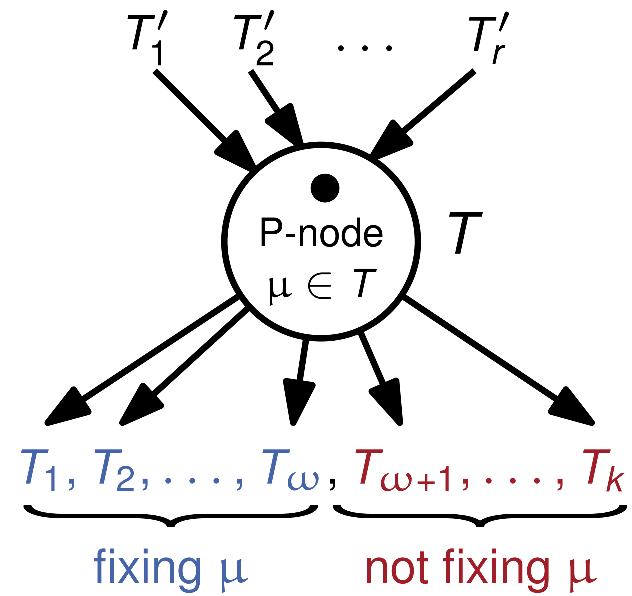
$I(G)$ can be tested in P-time ...

Simultaneous PQ-Ordering: Fixedness



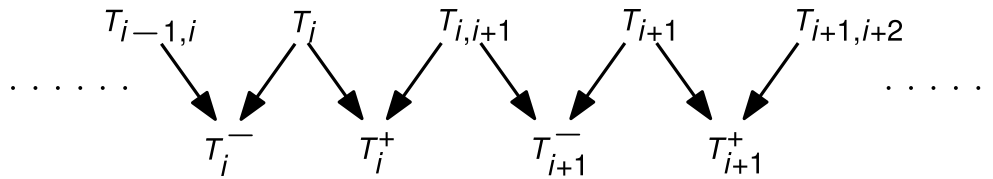
P-node μ' in T' *(parent)* is **fixed by** node μ in T *(child)* if there exist vertex-disjoint paths:

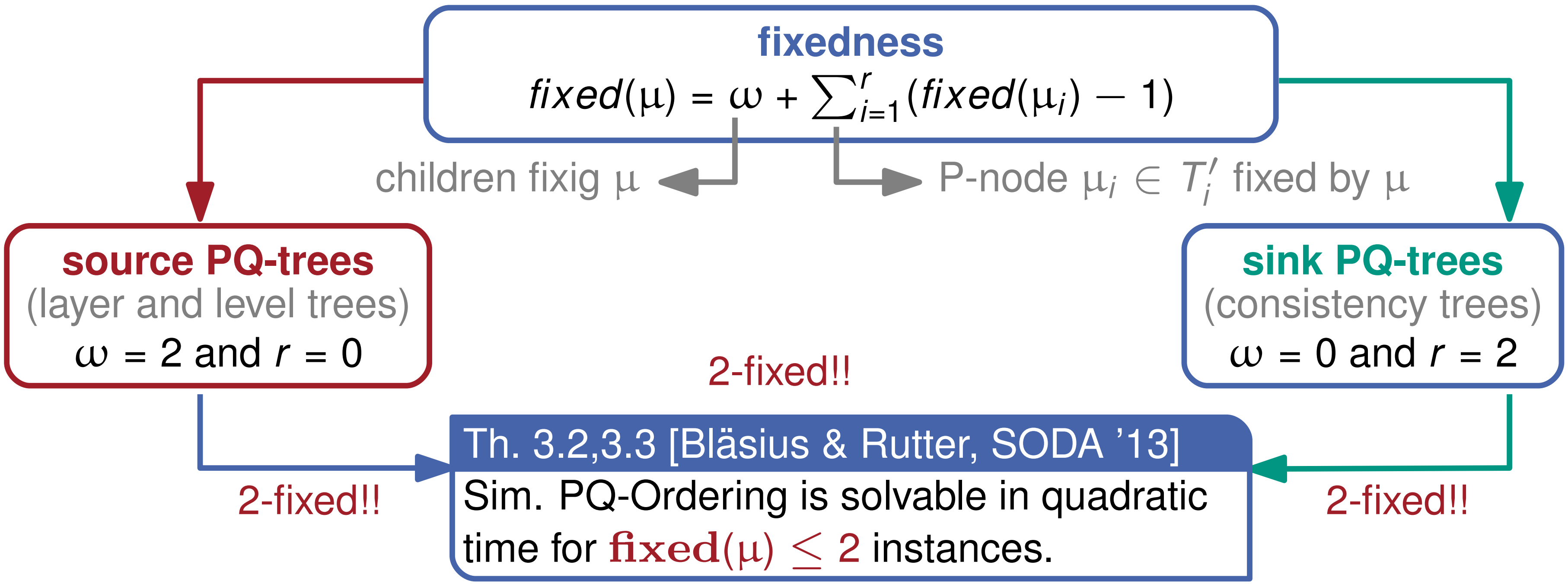
1. $a \to \mu$, $b \to \mu$, $c \to \mu$ in T and
2. $\phi(a) \to \mu'$, $\phi(b) \to \mu'$, $\phi(c) \to \mu'$ in T'



- some nodes (ω) in a children might fix a node in the parent
- **normalization**: we can assume that all nodes in a children fix **exactly a node** in the parent

Simultaneous PQ-Ordering: Fixedness





Theorem

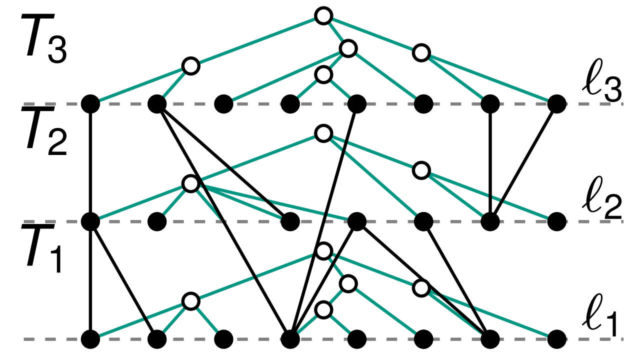
Torus Level Planarity (**Cyclic Level Planarity**) can be decided in $O(|V|^2)$ time for proper level graphs ($O(|V|^4)$ time for general level graphs)

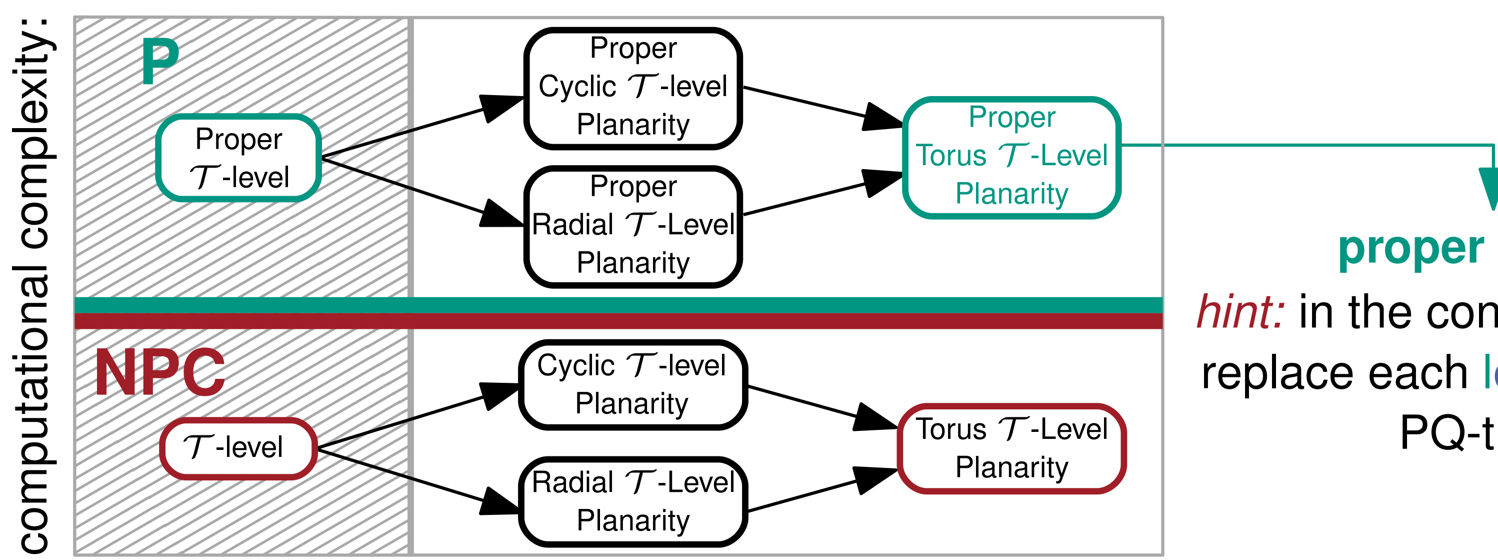
Cyclic, Radial, and Torus $\mathcal{T}$-Level Planarity

- **input:** level graph $(\bigcup_{i=1}^{k} V_i, E, \gamma)$ and set $\mathcal{T} = \bigcup_{i=1}^{k} \overline{T_i}$ of PQ-trees with $\mathcal{L}eaves(\overline{T_i}) = V_i$
- **question:** $\exists$ (Cyclic) Torus Level Embedding Γ in which $\forall_i$ the order $\mathcal{O}_i$ on V_i along ℓ_i is **consistent with** T_i?

Problem already studied in the **plane**:

- Wotzlaw, A., Speckenmeyer, E., Porschen, S.: **Generalized k-ary tanglegrams on level graphs: A satisfiability-based approach and its evaluation.** *DAM, 2012.*
- Angelini, P., Da Lozzo, G., Di Battista, G., Frati, F., Roselli, V.: **The importance of being proper: (in clustered-level planarity and T-level planarity).** *TCS, 2015.*





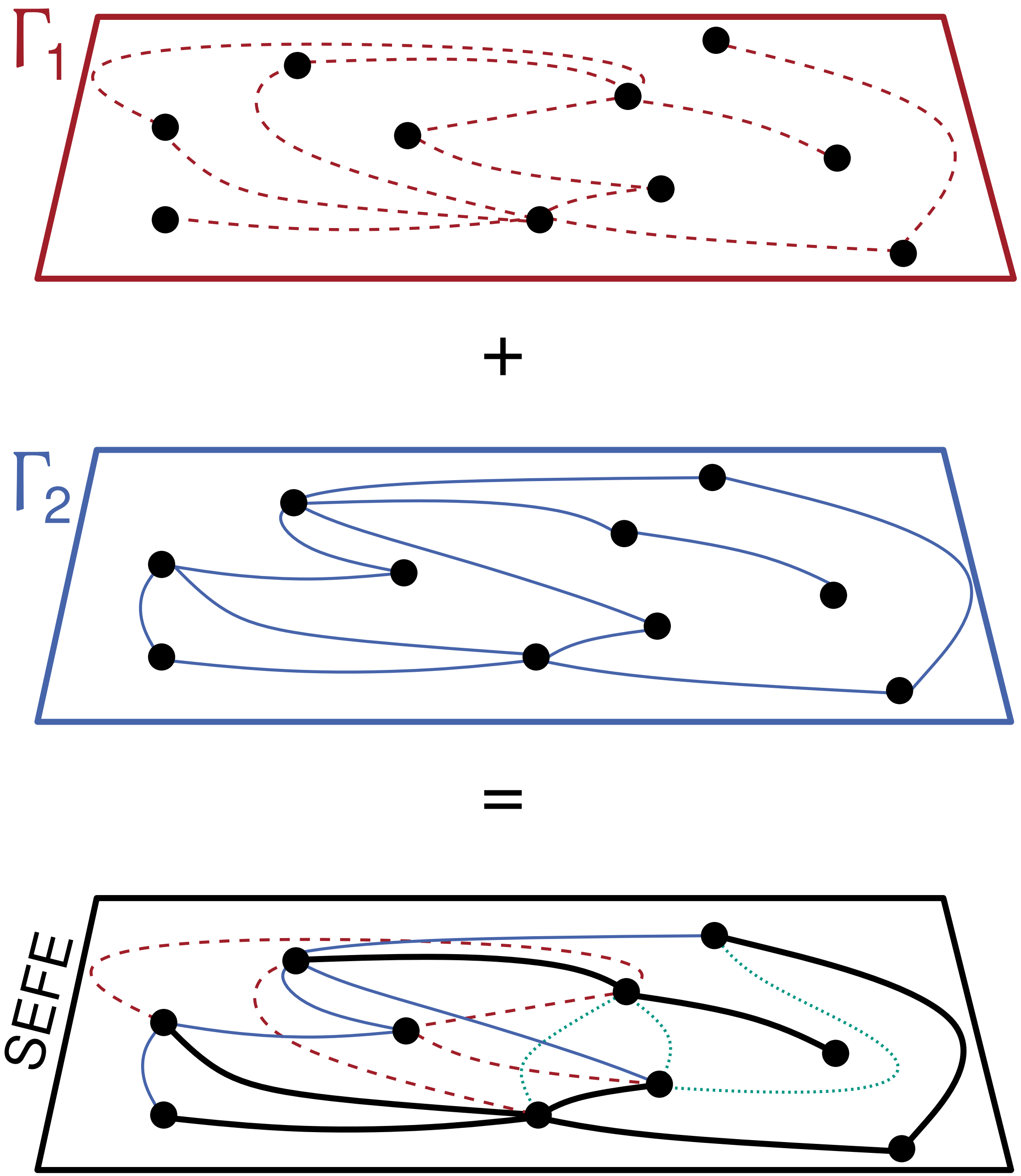
proper Instaces

hint: in the construction of $I(G)$ replace each level tree T_i with PQ-tree $\overline{T}_i$

Drawing Multiple Level Planar Graphs

(Simultaneous Level Planarity)

Simultaneous Embedding with Fixed Edges



Problem definition

- **input**: $G_1(V, E_1)$, $G_2(V, E_2)$
- **question:** existence of *planar drawings* Γ_1 and Γ_2 <u>such that</u>

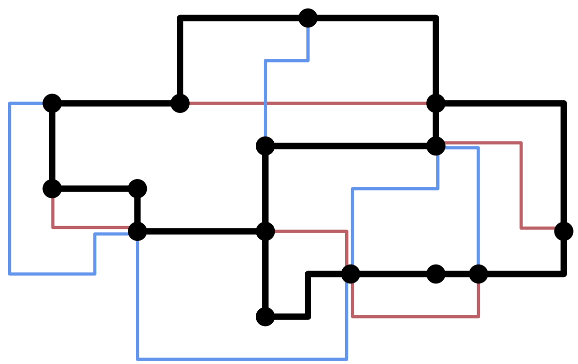
1. **vertices:**
   - $\forall v \in V, \Gamma_i(v) = \Gamma_j(v)$
2. **shared edges:**
   - $\forall e \in E_i \cap E_j, \Gamma_i(e) = \Gamma_j(e)$

Combining Problems

NEW PROBLEMS:= $\texttt{SEFE} + \texttt{Drawing Style}\ \mathcal{S}$

Examples

1. $\mathcal{S}$ = $\texttt{Orthogonal Drawings}$ $\rightarrow$ Angelini, P., Chaplick, S., Cornelsen, S., Da Lozzo, G., Di Battista, G., Eades, P., Kindermann, P., Kratochvìl, J., Lipp, F., Rutter, I.. **Simultaneous Orthogonal Planarity.** *GD '16.*



OrthoSEFE of $\langle (V, E_1), (V, E_2) \rangle$

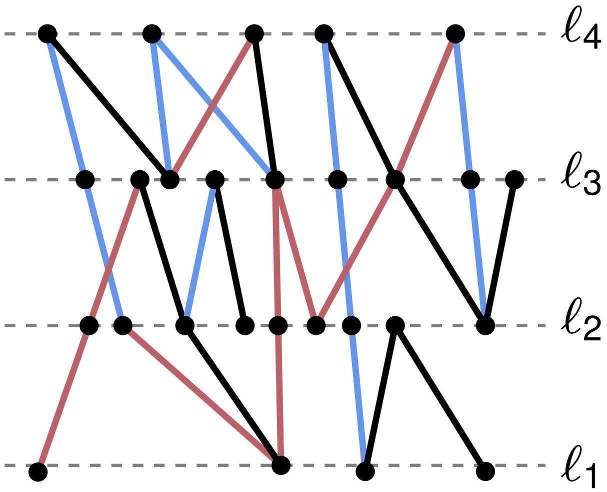
- $\mathcal{S} \in \{$$\texttt{Upward Drawings}$, $\texttt{Convex Drawings}$, **Level Drawings**, $\ldots\}$?

Simultaneous Level Planarity

Problem definition:

input: h proper level graphs $(V, E_1, \gamma), \ldots, (V, E_h, \gamma)$ (γ is the same $\forall$ i)

question: are there level embeddings Γ_i for each (V, E_i, γ) mapping the vertices in V to the same points along the corresponding levels?

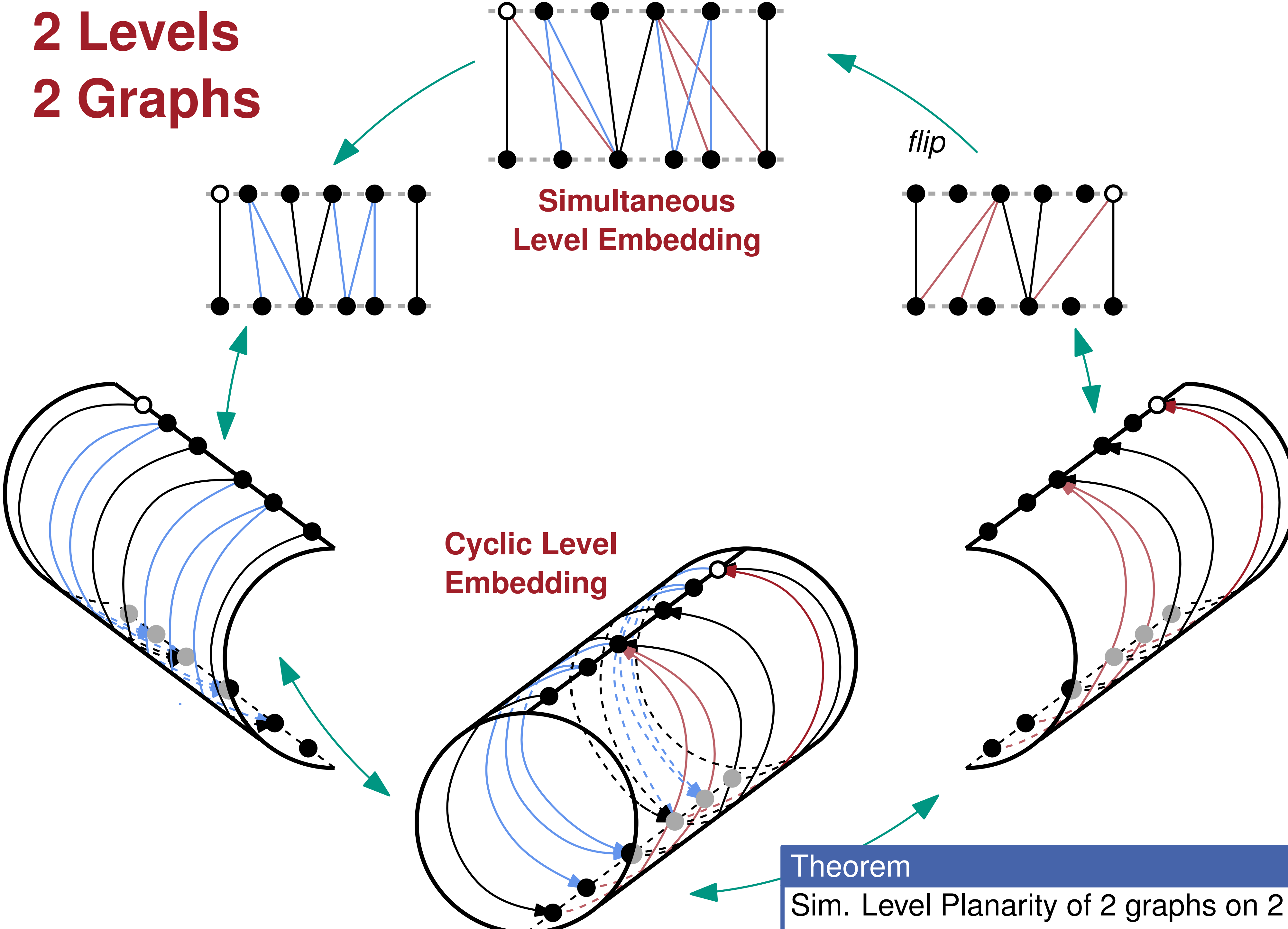


Sim. Level Embedding of $\langle(V, E_1, \gamma), (V, E_2, \gamma)\rangle$

Complexity

# Graphs \ # Levels	2	≥ 3
2	P	NPC
≥ 3	NPC	NPC

2 Levels
2 Graphs

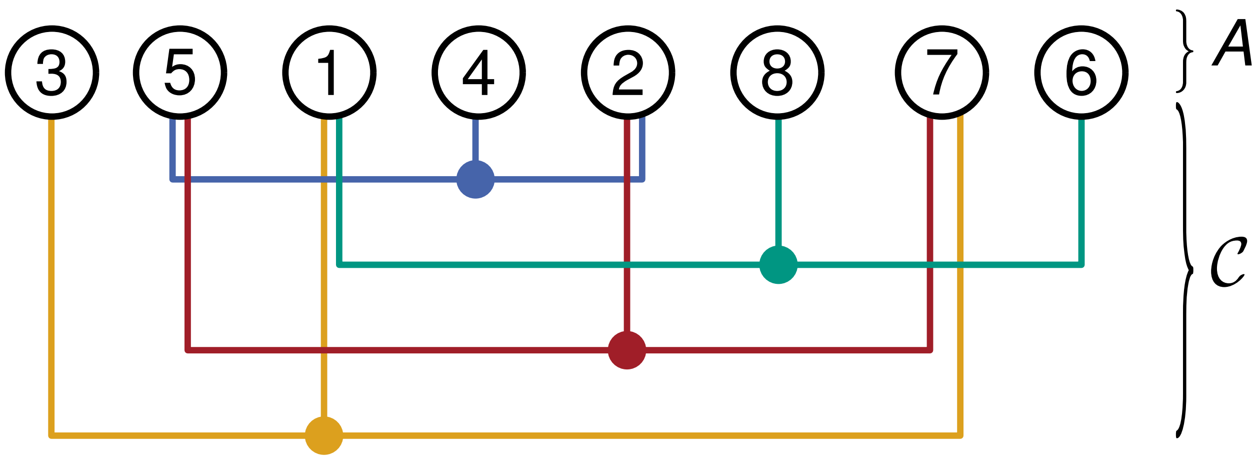


Theorem

Sim. Level Planarity of 2 graphs on 2 levels can be tested in quadratic time

The Betweenness Problem

- **input**: pair $\langle A, \mathcal{C} \rangle$
  - ∘ a finite set A of n objects
  - ∘ a set $\mathcal{C}$ of m ordered triples $t_i = \langle \alpha_i, \boldsymbol{\beta}_i, \delta_i \rangle$ of objects in A



$t_1 = \langle 5, 4, 2 \rangle$

$t_2 = \langle 1, 8, 6 \rangle$

$t_3 = \langle 7, 2, 5 \rangle$

$t_4 = \langle 7, 1, 3 \rangle$

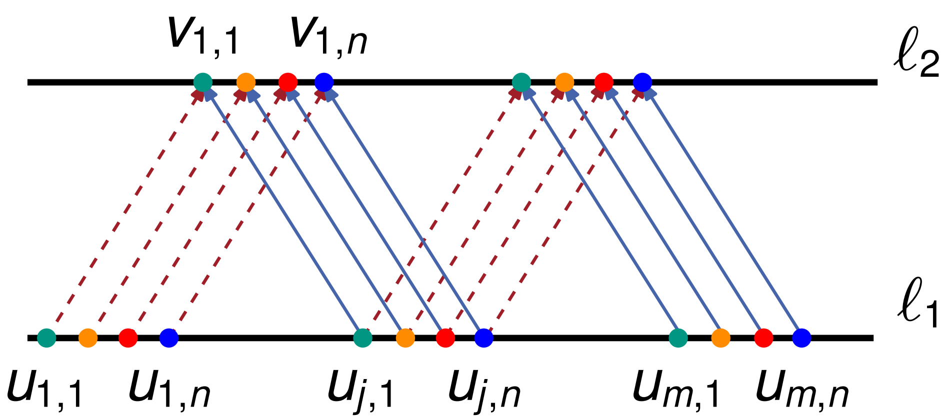
- **question**: existence of a **linear ordering** $\mathcal{O}$ on A such that, for each triple $t_i \in \mathcal{C}$, it holds either

$$\mathcal{O} = \langle \ldots, \alpha_i, \ldots, \boldsymbol{\beta}_i, \ldots, \delta_i, \ldots \rangle \text{ or } \mathcal{O} = \langle \ldots, \delta_i, \ldots, \boldsymbol{\beta}_i, \ldots, \alpha_i, \ldots \rangle$$

Theorem [Opatrny - J. Comp.'79]
Betweenness is $\mathcal{NP}$-Complete

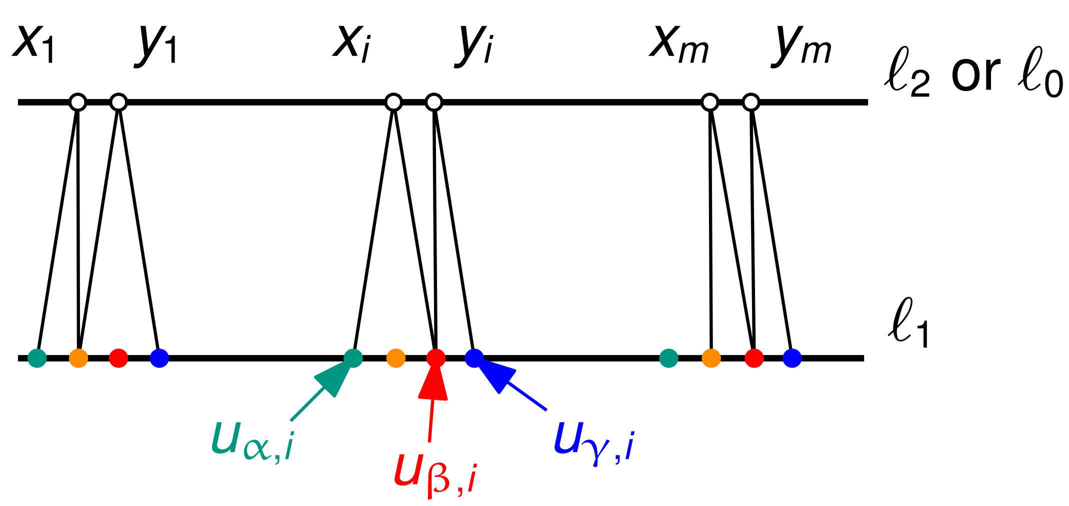
Gadgets

Ordering gadget: pair $\langle (V, E_1, \gamma), (V, E_2, \gamma) \rangle$ of level graphs on two levels



Property:
$\forall i = 1, \ldots, |A|$, vertices
$u_{i,1}, \ldots, u_{i,n}$
appear in the same left-to-right order

Triplet gadget: single level graph (V, E, γ) on two levels

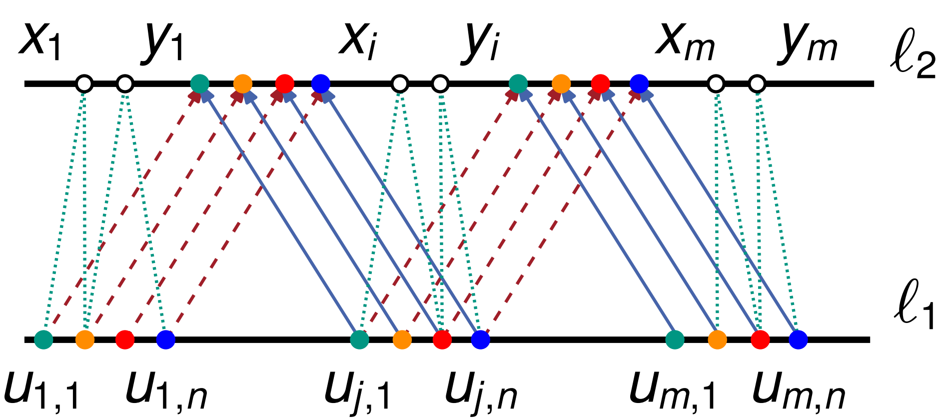


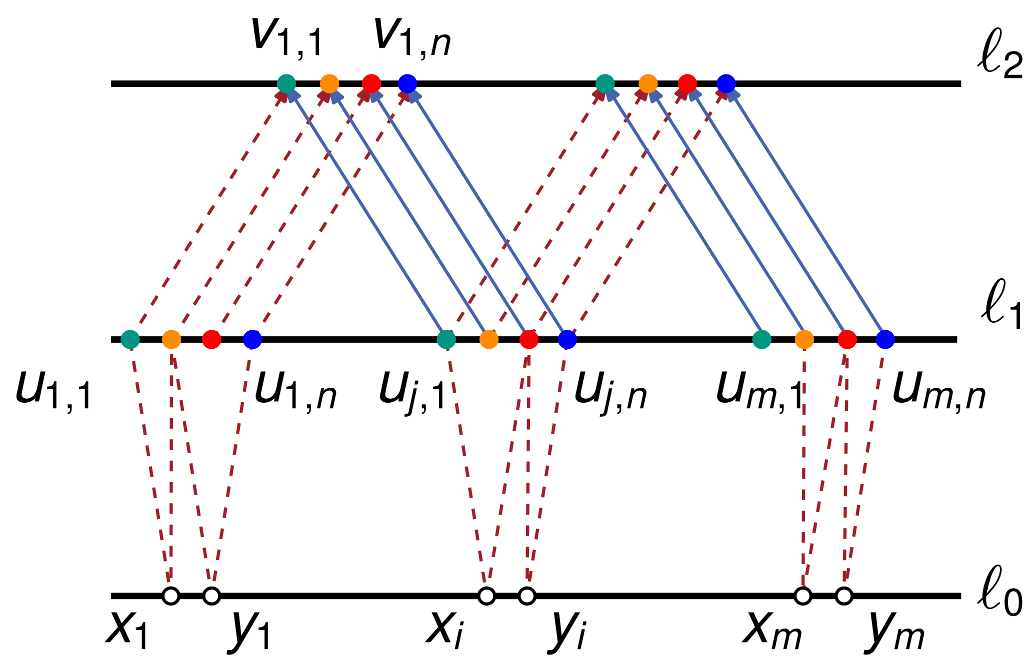
Property:
$\forall i = 1, \ldots, |\mathcal{C}|$, vertex $u_{\beta,j}$
is between $u_{\alpha,j}$ and $u_{\gamma,j}$
along ℓ_1

NP-completeness

Theorem

Simultaneous Level Planarity is *NP*-complete even for
3 graphs on 2 levels and **2 graphs on 3 levels**





- all graphs can be made **connected** at the expense of using one **additional level**
- all theorems hold true even if the simultaneous embedding is **geometric**, **without fixed edges**, or **with fixed edges**

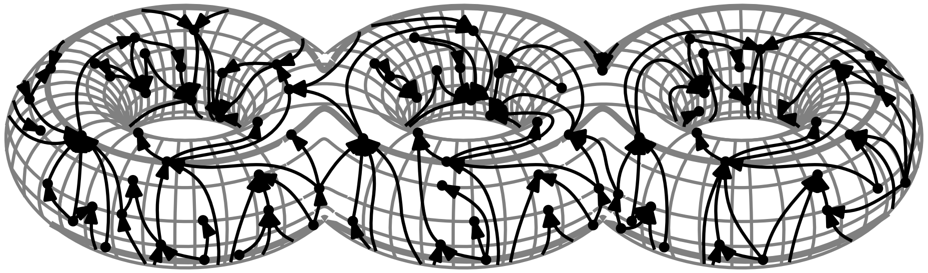
Conclusions and Open Problems

Results:

- we gave a simple testing and embedding algorithm for TORUS LEVEL PLANARITY (CYCLIC LEVEL PLANARITY) that runs in $O(|V|^2)$ time for proper level graphs ($O(|V|^4)$ time for general level graphs)
- we established a complexity dichotomy for SIMULTANEOUS LEVEL PLANARITY w.r.t. the # of graphs and the # of levels

Open problems:

- design new techniques to improve the time bounds
- extend the concept of level planarity to surfaces of higher genus



EBERHARD KARLS UNIVERSITÄT TÜBINGEN · KIT Karlsruhe Institute of Technology

Beyond Level Planarity

24th International Symposium on Graph Drawing & Network Visualization

19–21 September, Athens, Greece

Patrizio Angelini, **Giordano Da Lozzo**, Fabrizio Frati,
Giuseppe Di Battista, Maurizio Patrignani, Ignaz Rutter

WILHELM-SCHICKHARD-INSTITUT FÜR INFORMATIK · TÜBINGEN UNIVERSITY
DEPARTMENT OF ENGINEERING · ROMA TRE UNIVERSITY FACULTY OF INFORMATICS · KARLSRUHE INSTITUTE OF TECHNOLOGY

Thank you!

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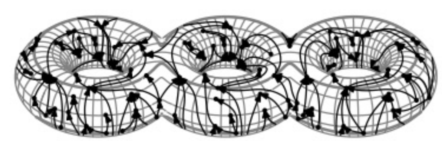

GD '16 – Beyond Level Planarity