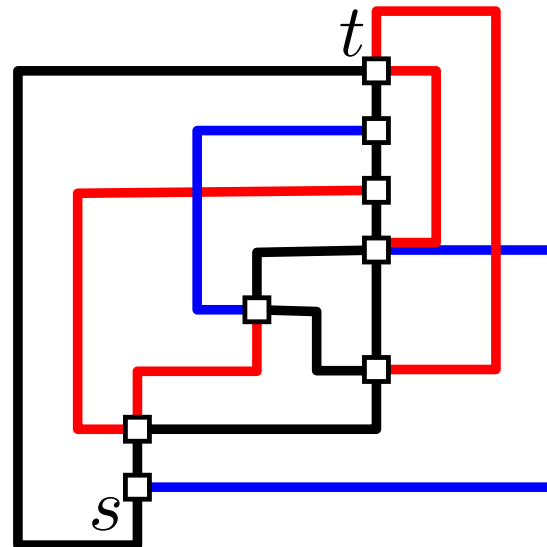
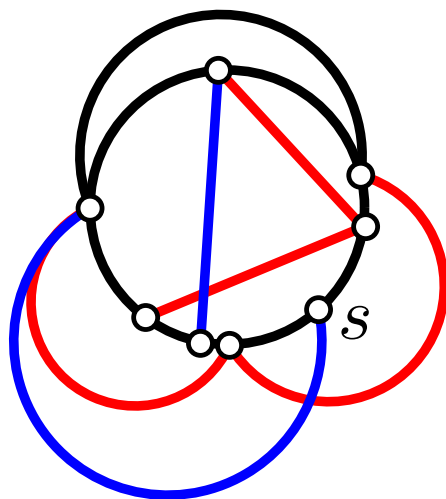




# Simultaneous Orthogonal Drawing

Patrizio Angelini, Steve Chaplick, Sabine Cornelsen,  
Giordano Da Lozzo, Giuseppe Di Battista, Peter Eades,  
**Philipp Kindermann**, Jan Kratochvíl, Fabian Lipp, Ignaz Rutter



# Simultaneous Drawing

Given two graphs  $G_1 = (V, E_1)$  and  $G_2 = (V, E_2)$ .

Are there drawings of  $G_1$  and  $G_2$  that coincide on  $G$ ?

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$G = G_1 \cap G_2$   
common graph

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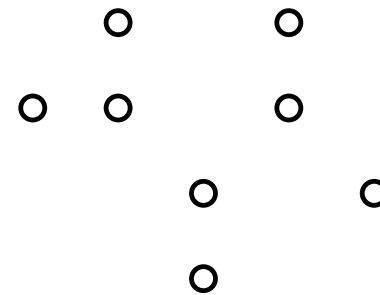
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It depends:

► planar, straight-line



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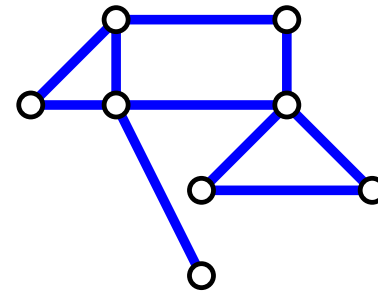
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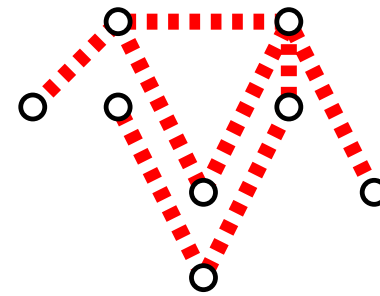
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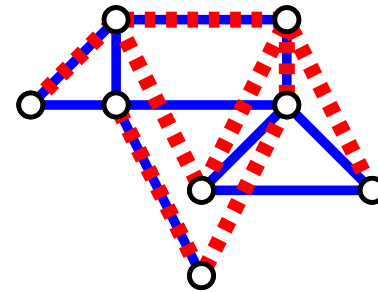
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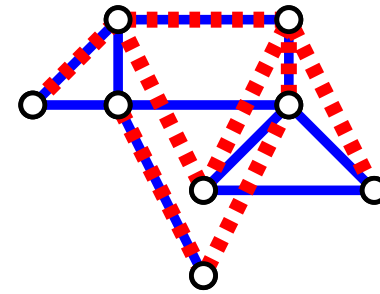
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► NP-hard [Estrella-Balderrama et al. '07]

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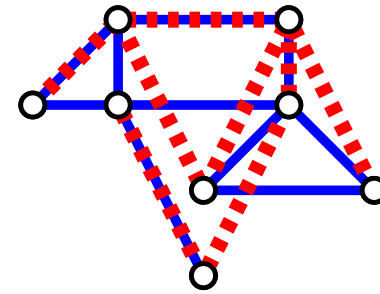
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- ▶ NP-hard [Estrella-Balderrama et al. '07]
- ▶ There exist a tree and a path that don't work [Angelini et al. '12]

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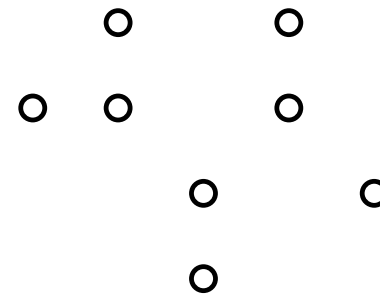
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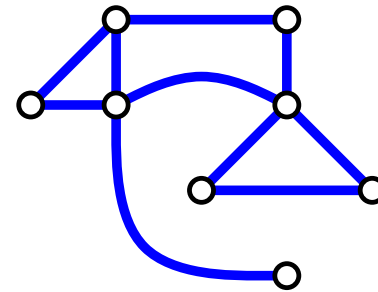
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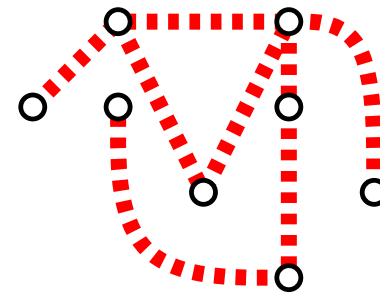
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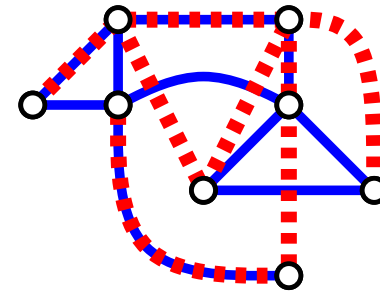
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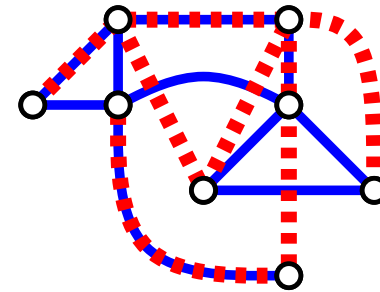
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Efficiently solvable if...

- ▶  $G$  is biconnected [Haeupler et al. '13]



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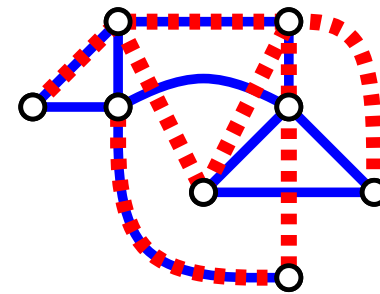
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Efficiently solvable if...

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- ▶  $G$  is a star [Angelini et al. '12]

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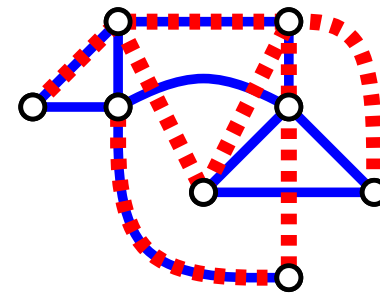
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Efficiently solvable if...

- ▶  $G$  is biconnected [Haeupler et al. '13]
- ▶  $G$  is a star [Angelini et al. '12]
- ▶  $G_1, G_2$  are biconnected,  
 $G$  is connected [Bläsius & Rutter '16]
- ▶ ...

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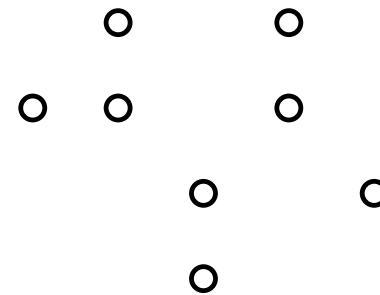
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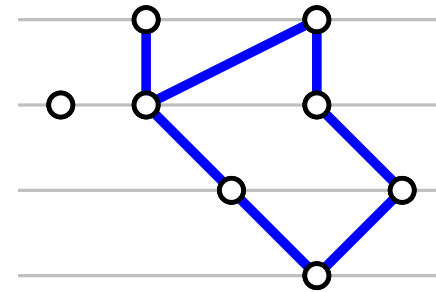
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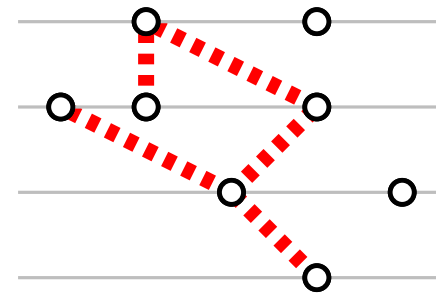
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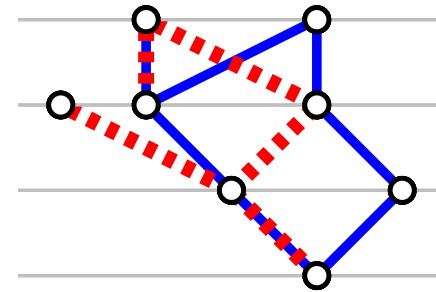
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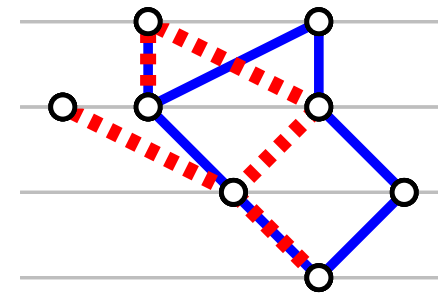
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- ▶ efficiently solvable  
for 2 levels

[Angelini et al. '16]

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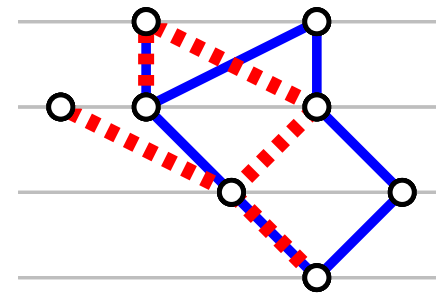
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- ▶ efficiently solvable for 2 levels [Angelini et al. '16]
- ▶ NP-hard for 3 levels [Angelini et al. '16]



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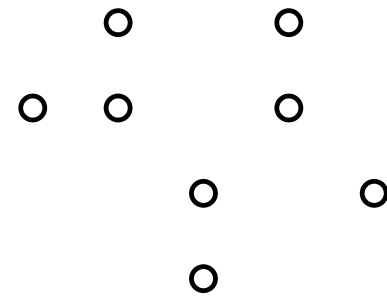
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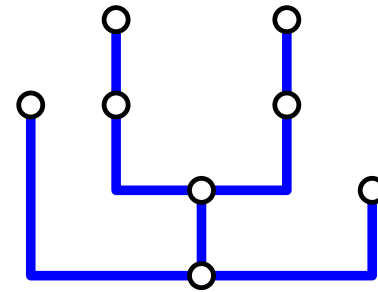
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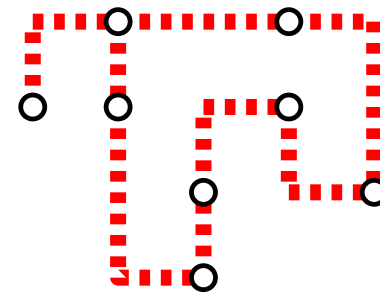
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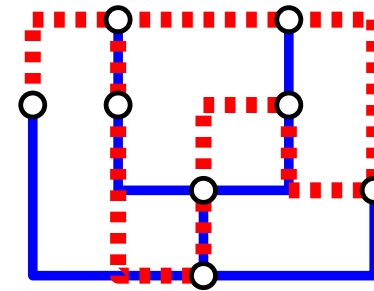
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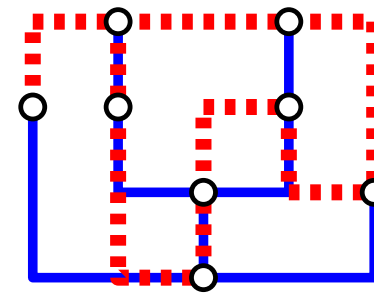
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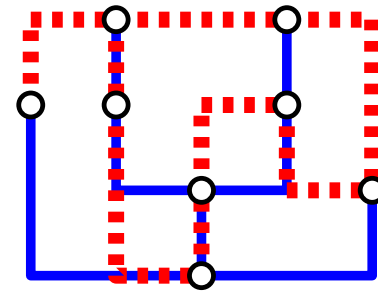
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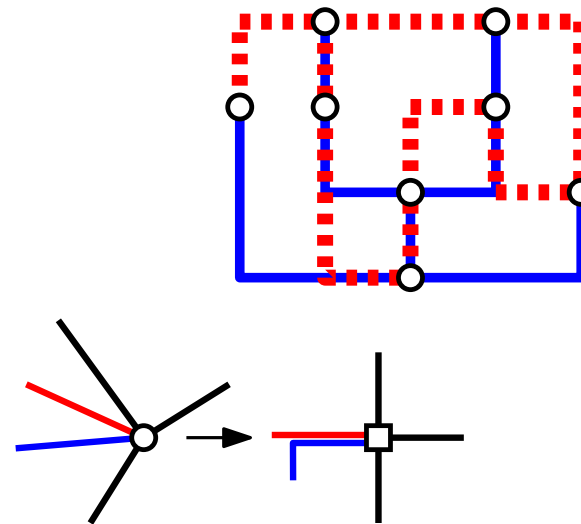
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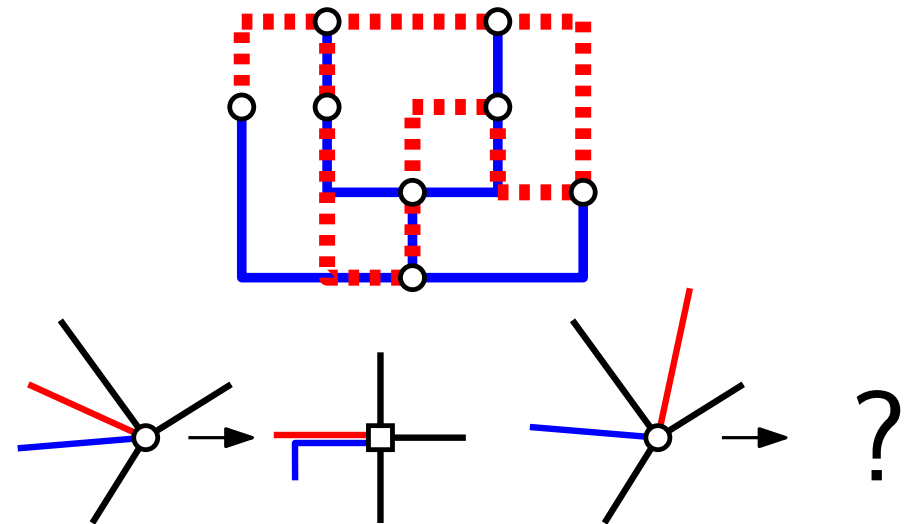
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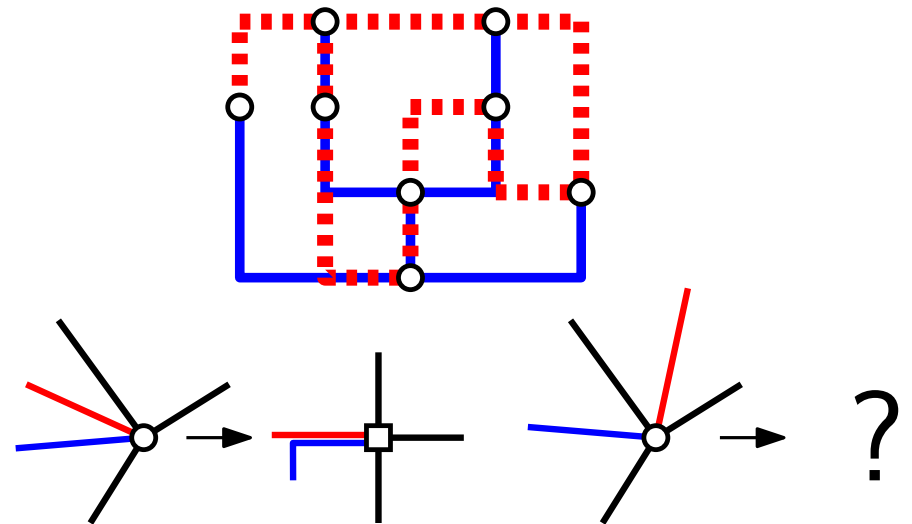
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Necessary Conditions:

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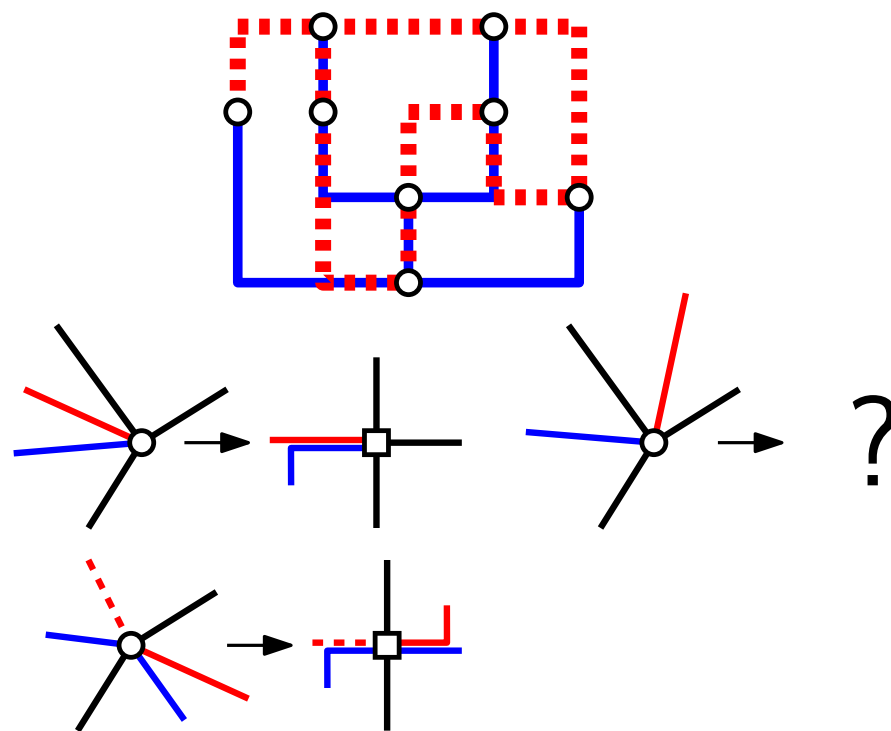
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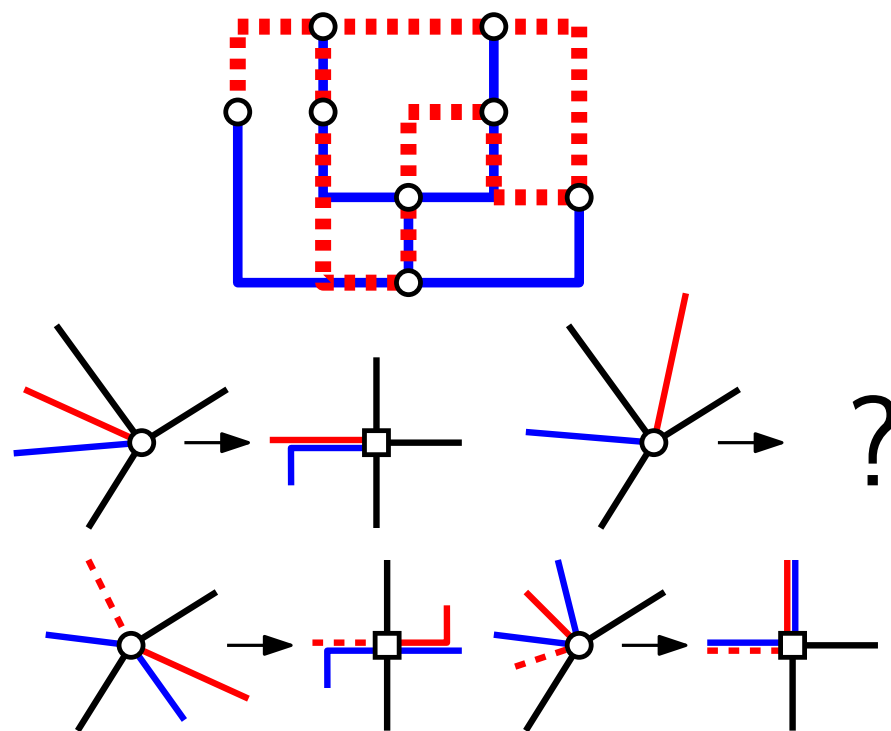
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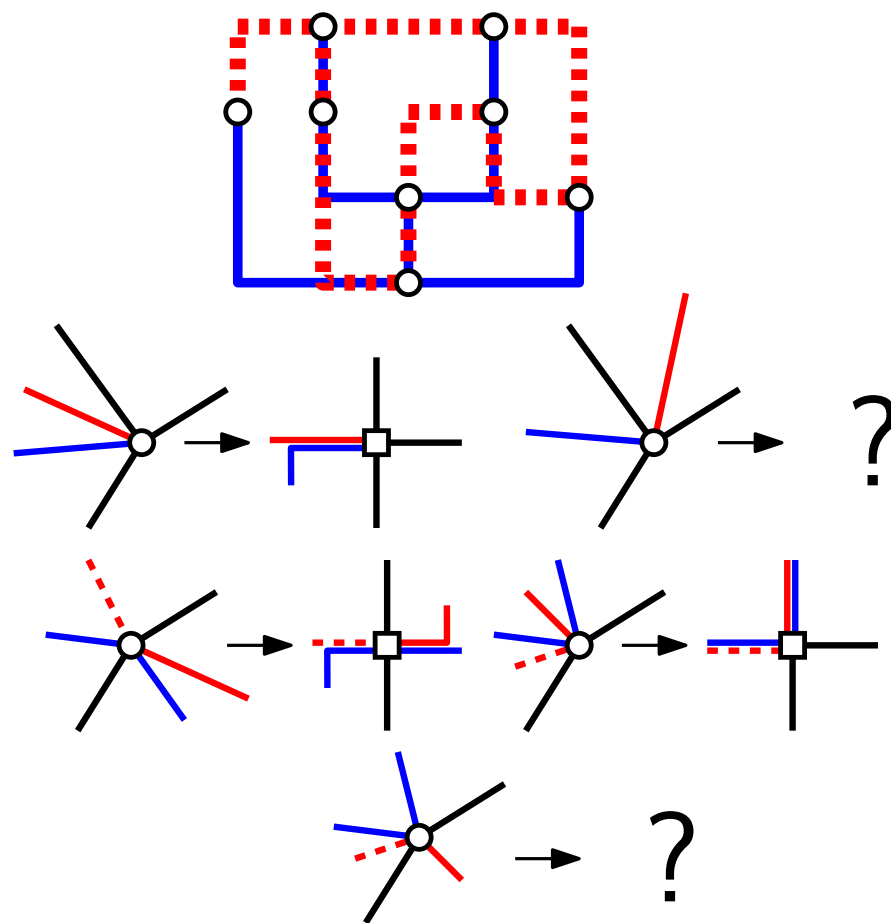
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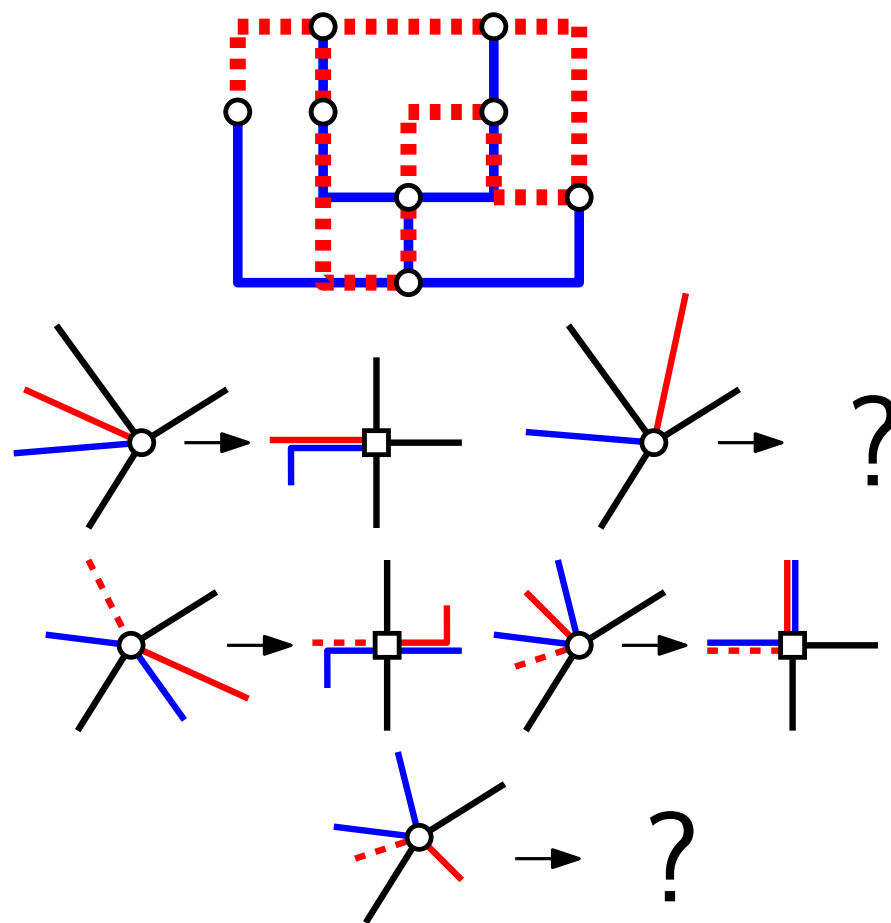
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Necessary Conditions:

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- ▶ embeddings allow orthogonal vertex drawings

Conditions are sufficient!



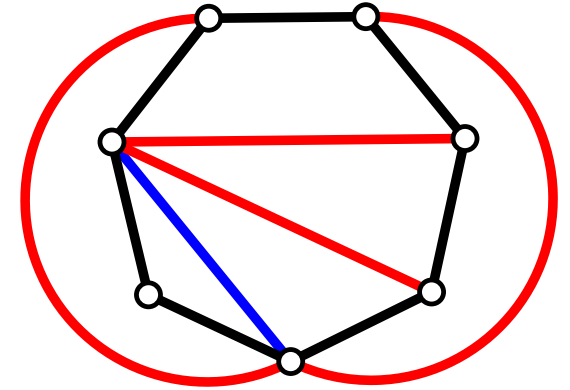
# Special Case: Common Graph is a Cycle

Isn't this trivial?

# Special Case: Common Graph is a Cycle

Isn't this trivial?

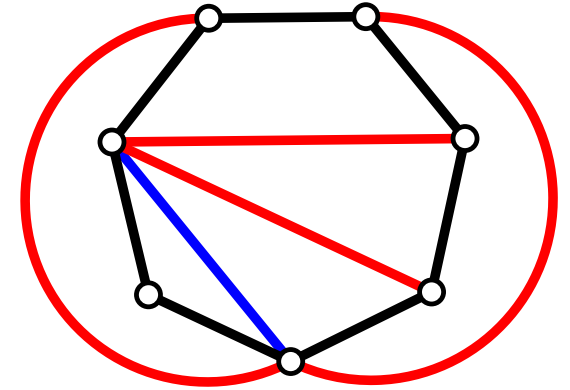
- ▶ Instances trivially admit a SEFE...



# Special Case: Common Graph is a Cycle

Isn't this trivial?

- ▶ Instances trivially admit a SEFE...
- ▶ but not necessarily an ORTHOSEFE.

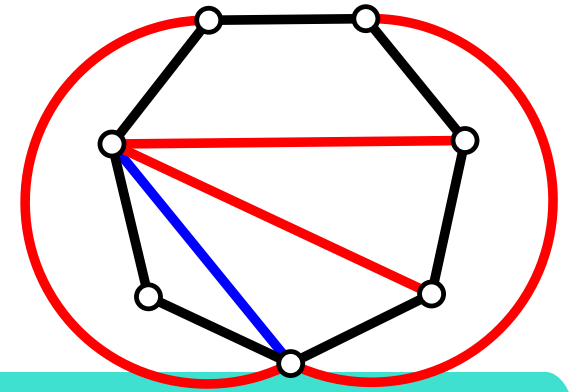




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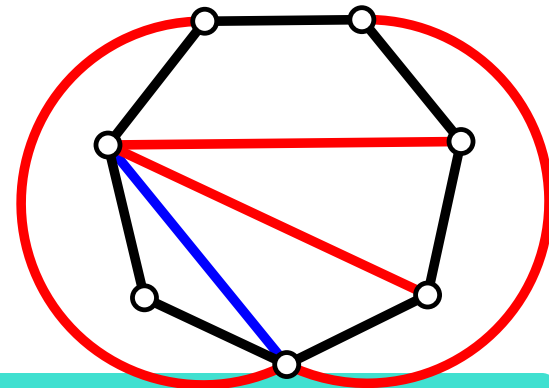
### Theorem.

It is NP-complete to decide whether three graphs  $G_1, G_2, G_3$  whose common graph is a cycle admit an ORTHOSEFE.

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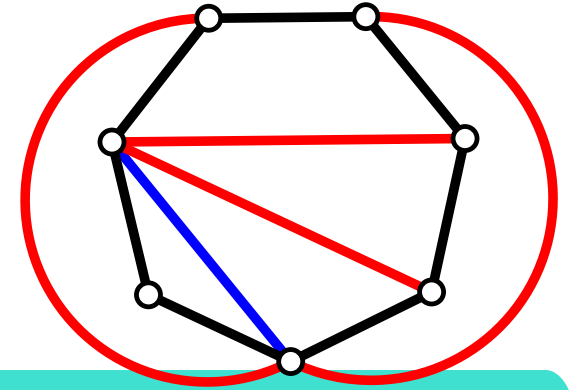
### Proof:

Reduction from NAE-3SAT:

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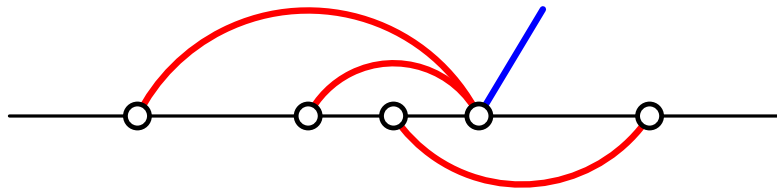
Reduction from NAE-3SAT:

Given:  $X$  set of variables,  $C$  set of clauses each containing 3 literals

Find: Truth assignment such that no clause in  $C$  evaluates to (TRUE, TRUE, TRUE) or (FALSE, FALSE, FALSE)

# Reduction from NAE-3SAT

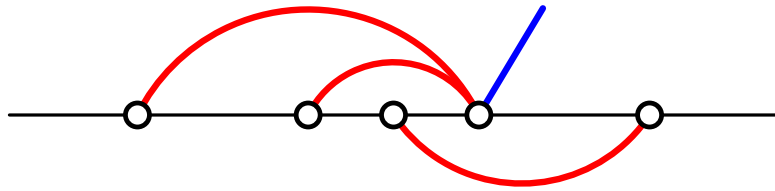
Variable gadget for variable  $x$ :



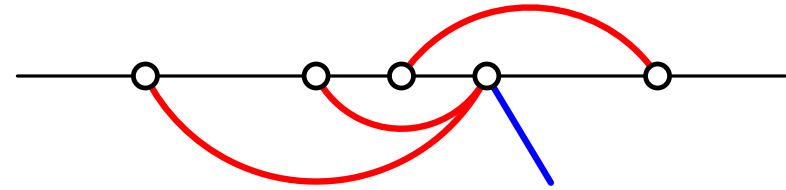
$x = \text{TRUE}$

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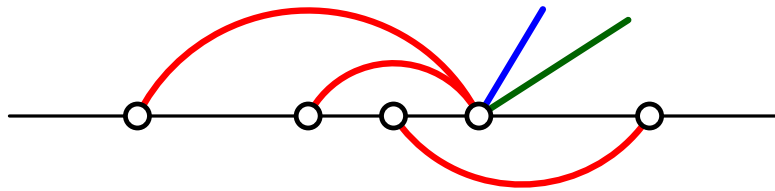
$x = \text{TRUE}$



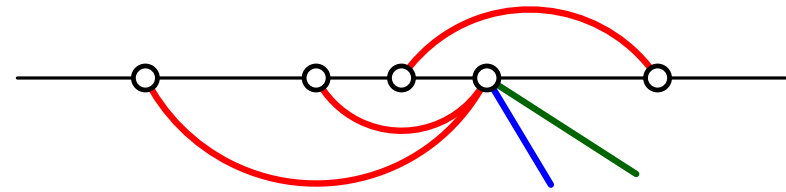
$x = \text{FALSE}$

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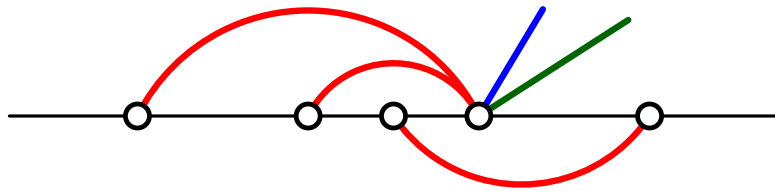
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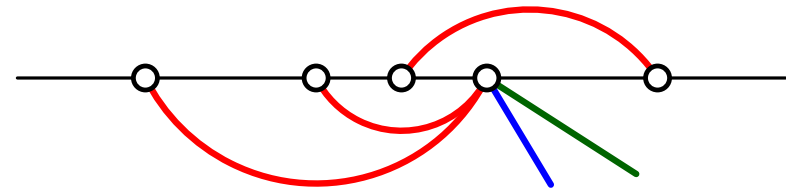
$x = \text{FALSE}$

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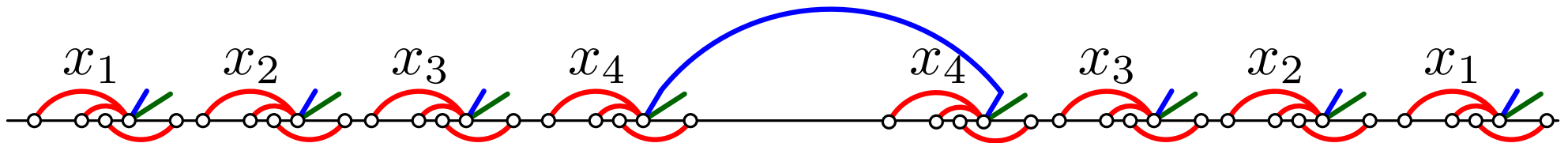
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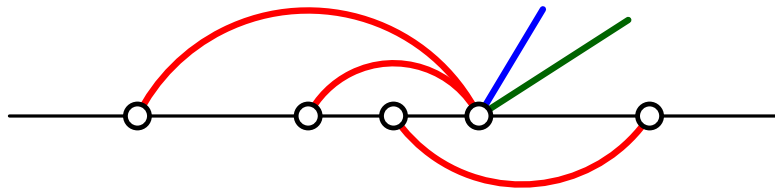


$x = \text{FALSE}$

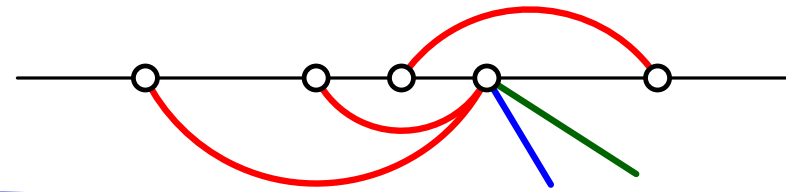


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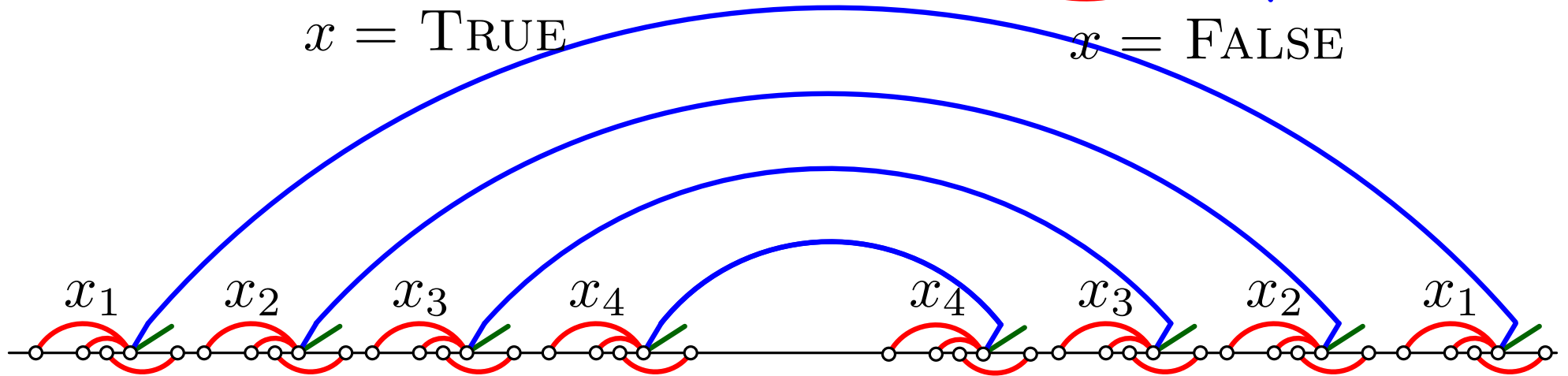
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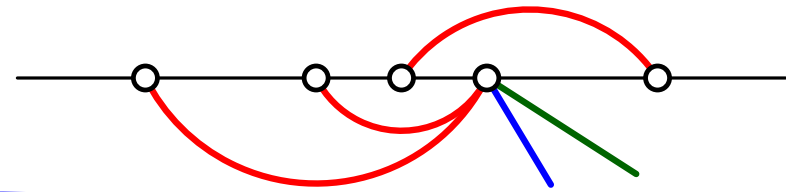
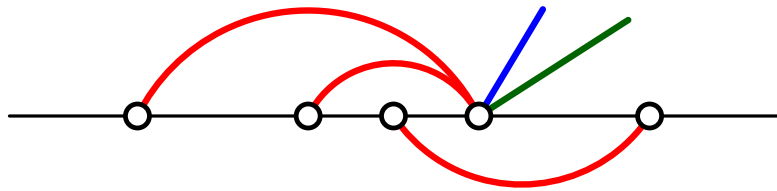
$x = \text{FALSE}$





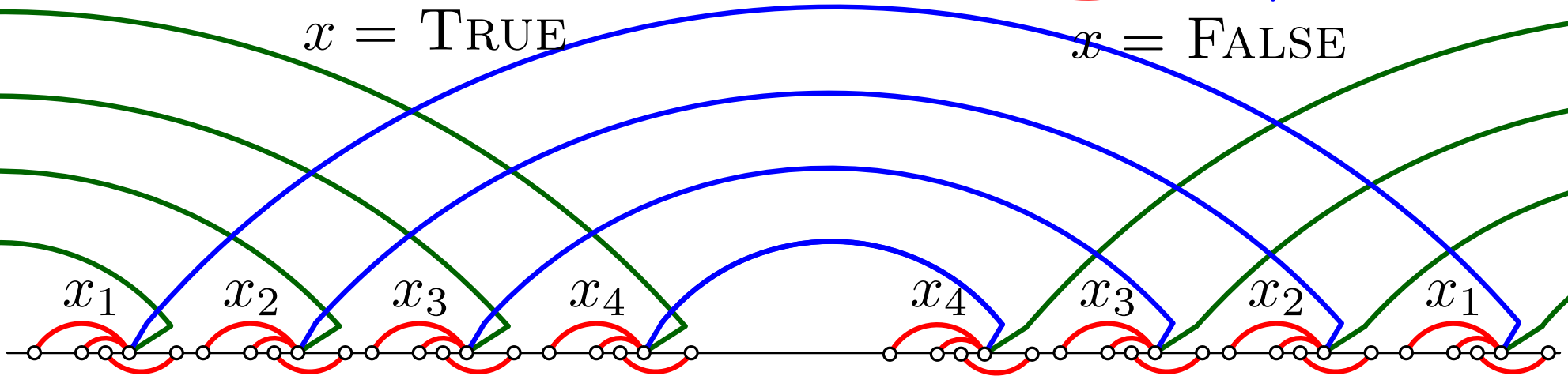
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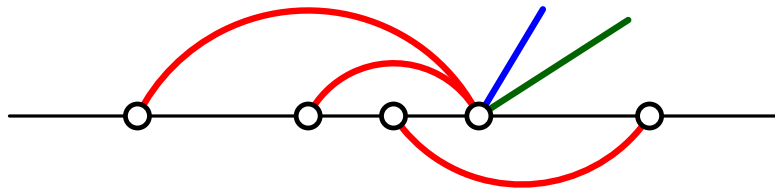
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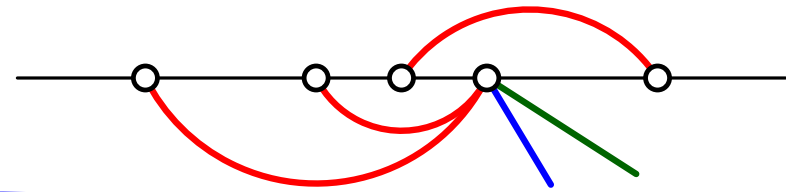


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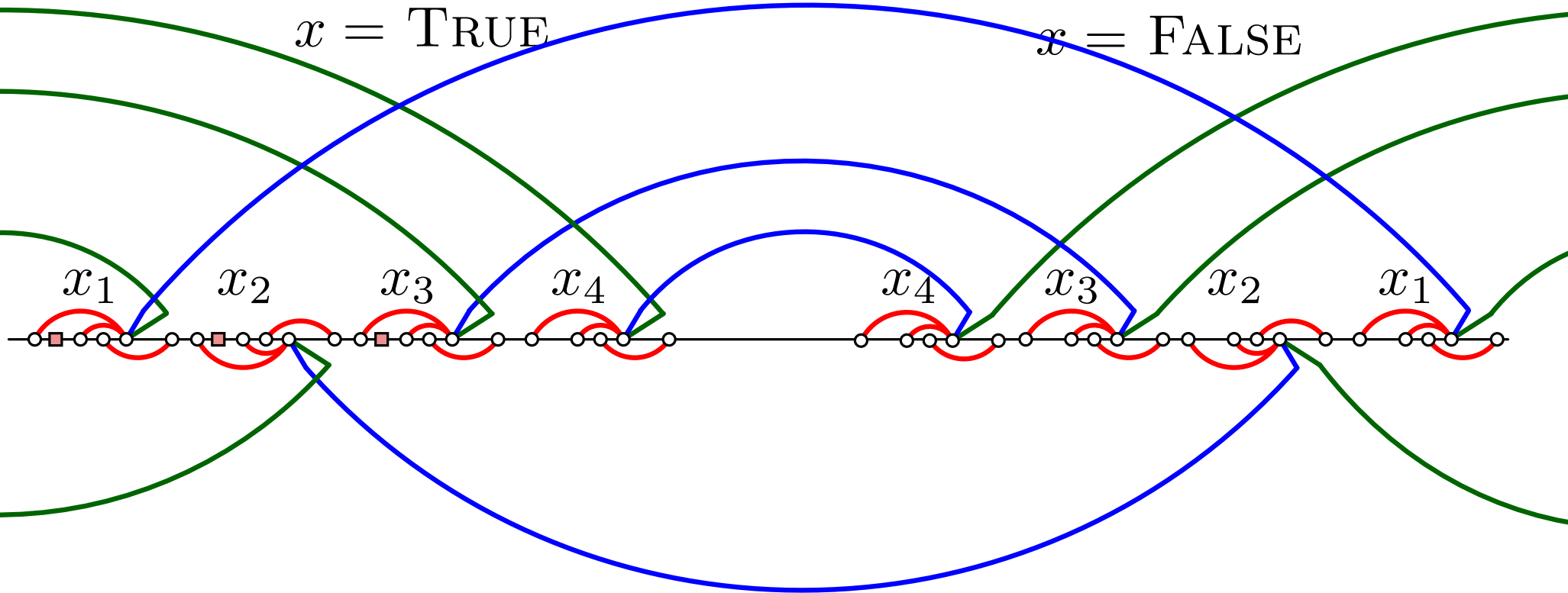
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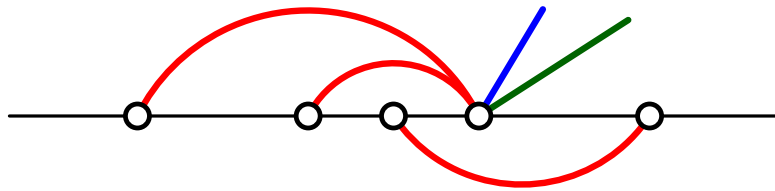


$x = \text{FALSE}$

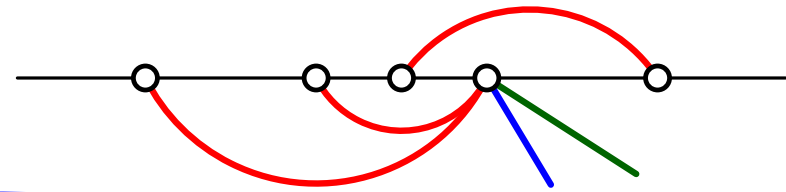


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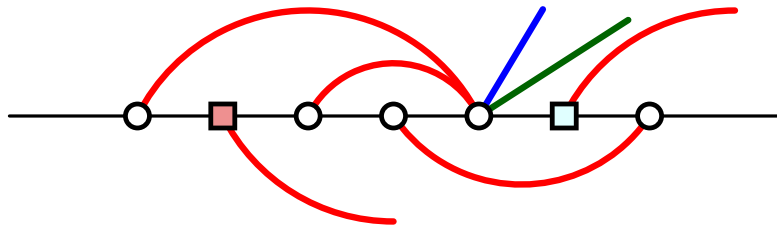


$x = \text{FALSE}$

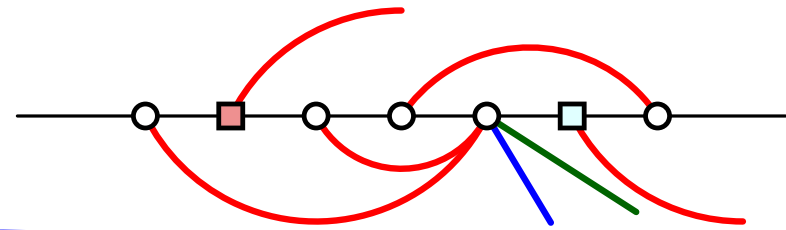


# Reduction from NAE-3SAT

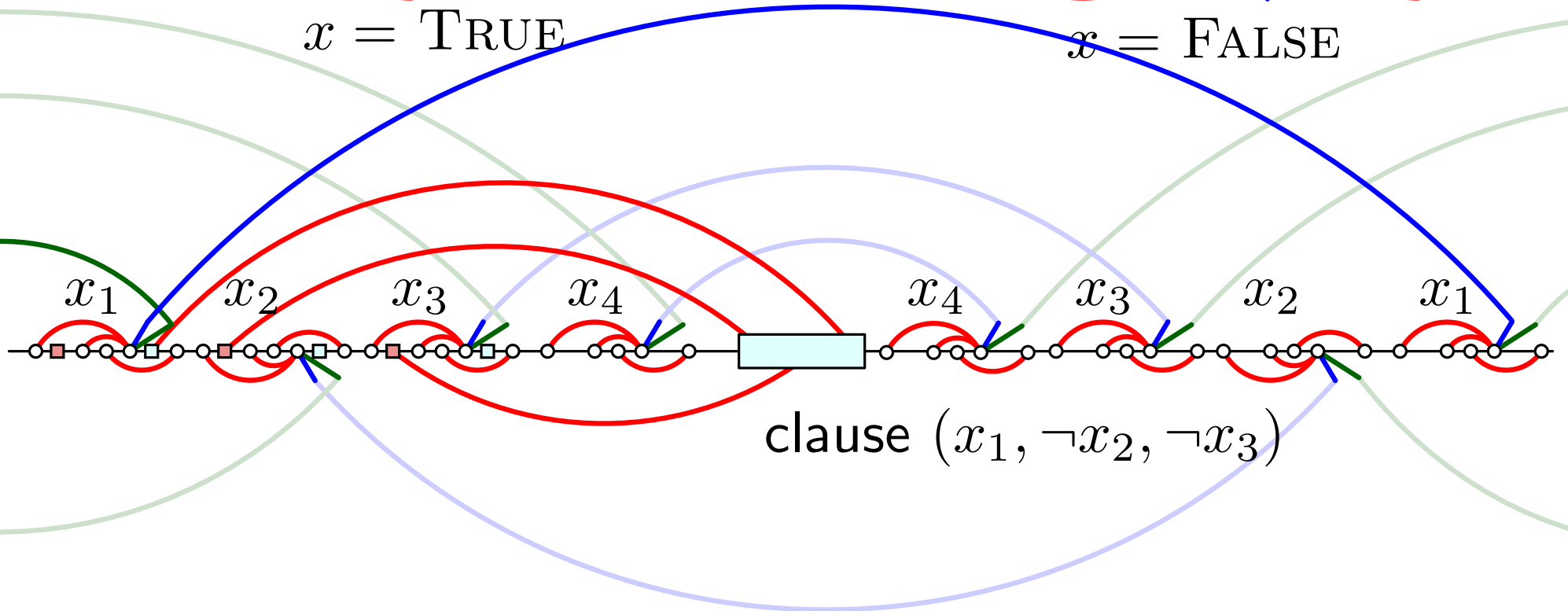
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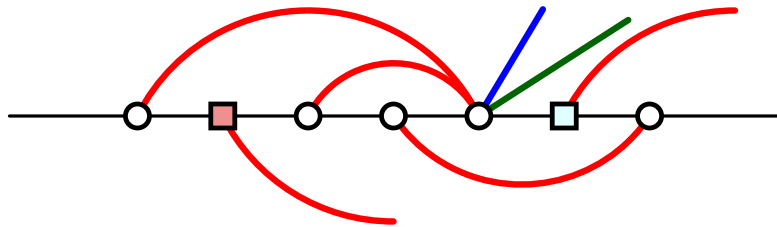
$x = \text{FALSE}$



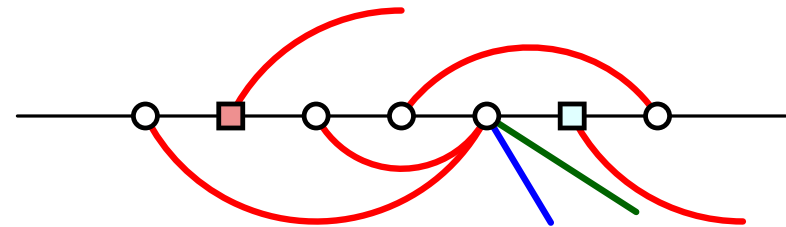
clause  $(x_1, \neg x_2, \neg x_3)$

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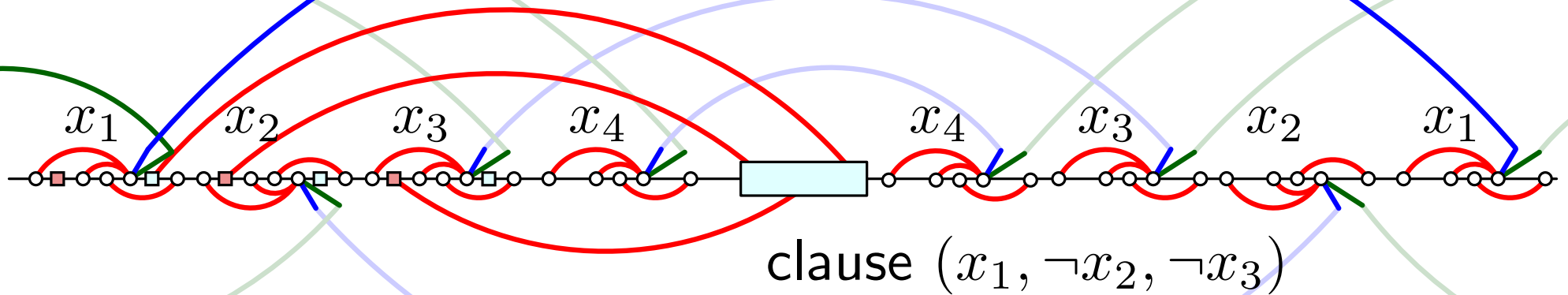
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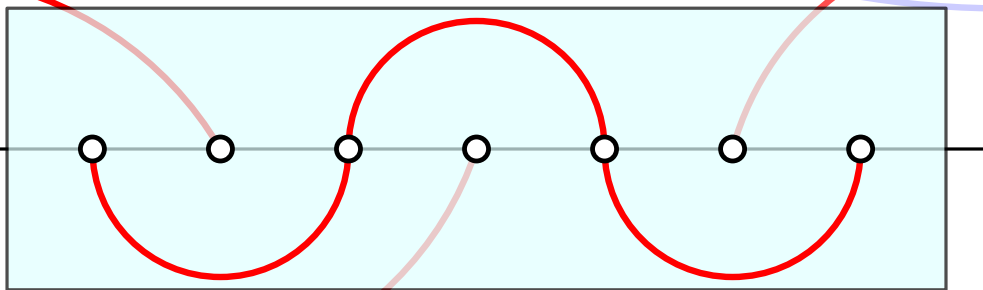
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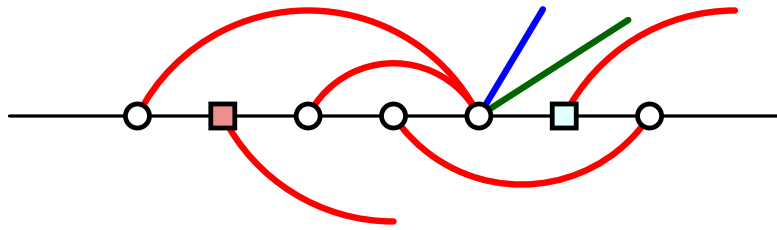




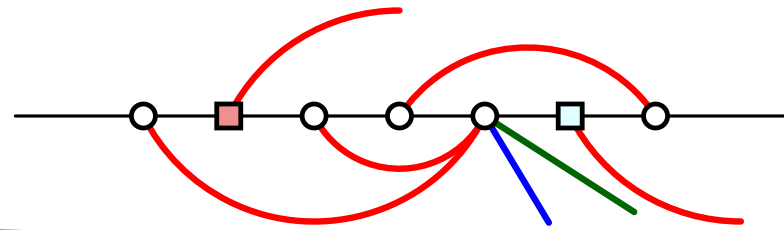


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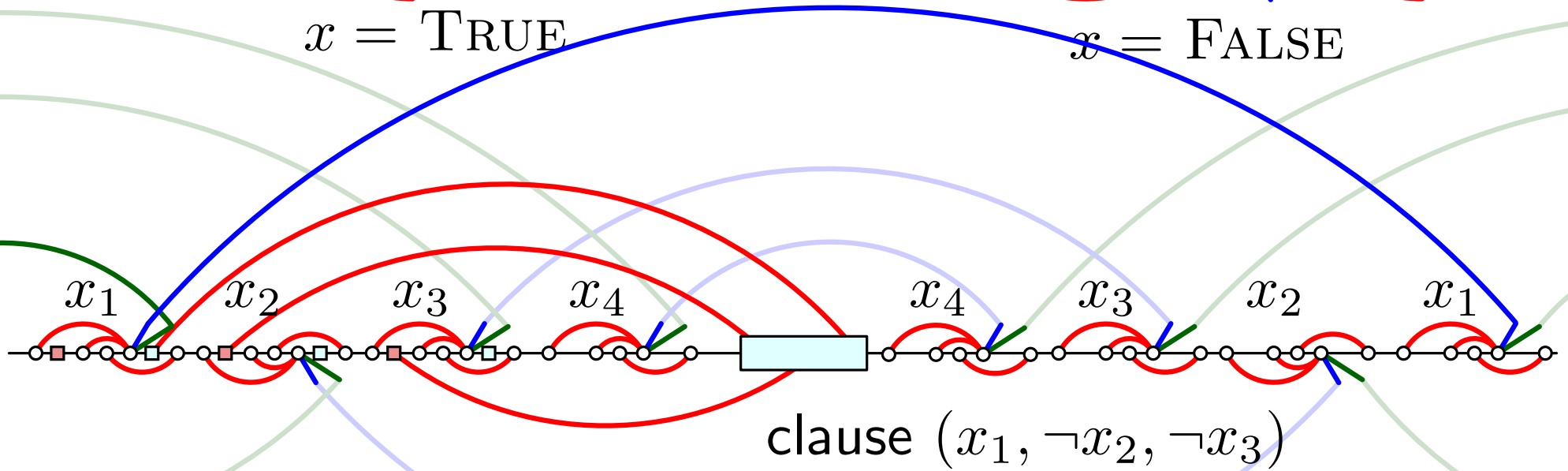
Variable gadget for variable  $x$ :



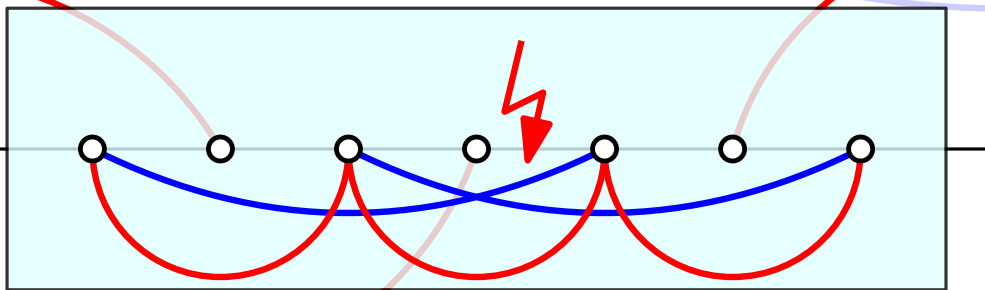
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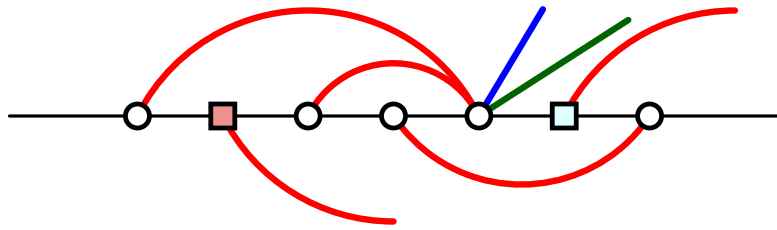
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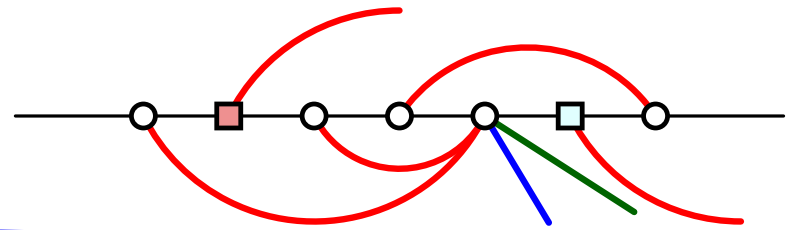


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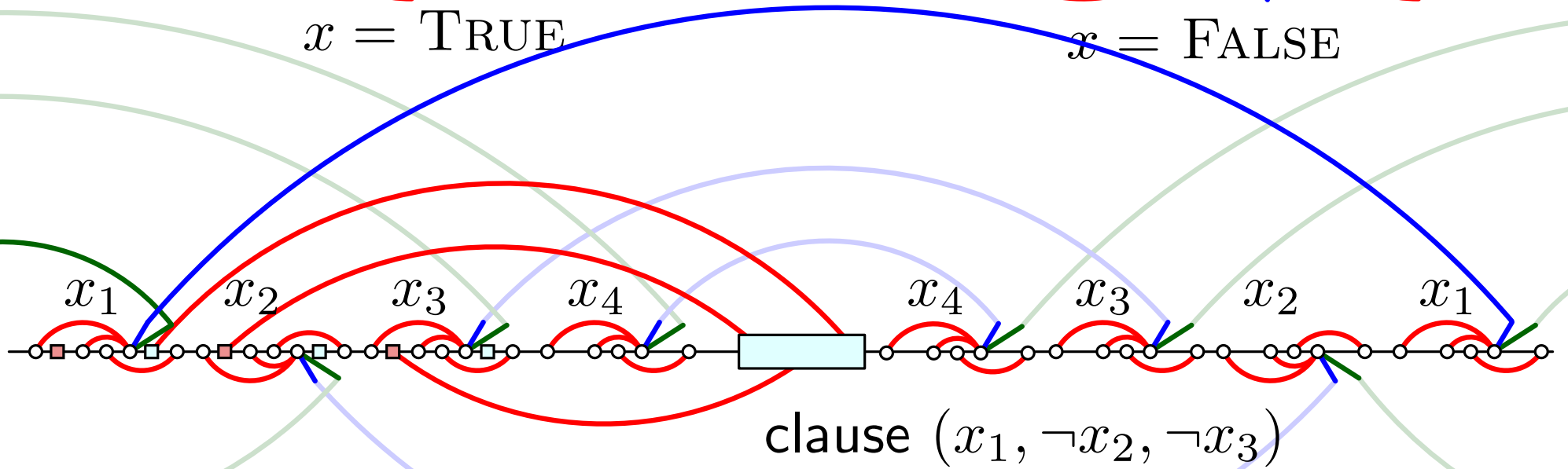
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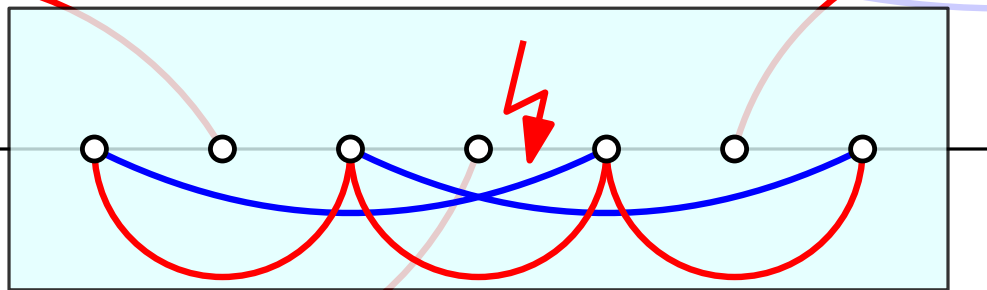
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Red edges on different sides  
 $\Leftrightarrow$   
Blue edges on different sides.



## Reduction from NAE-3SAT

Reduction also works for two colors:

## Reduction from NAE-3SAT

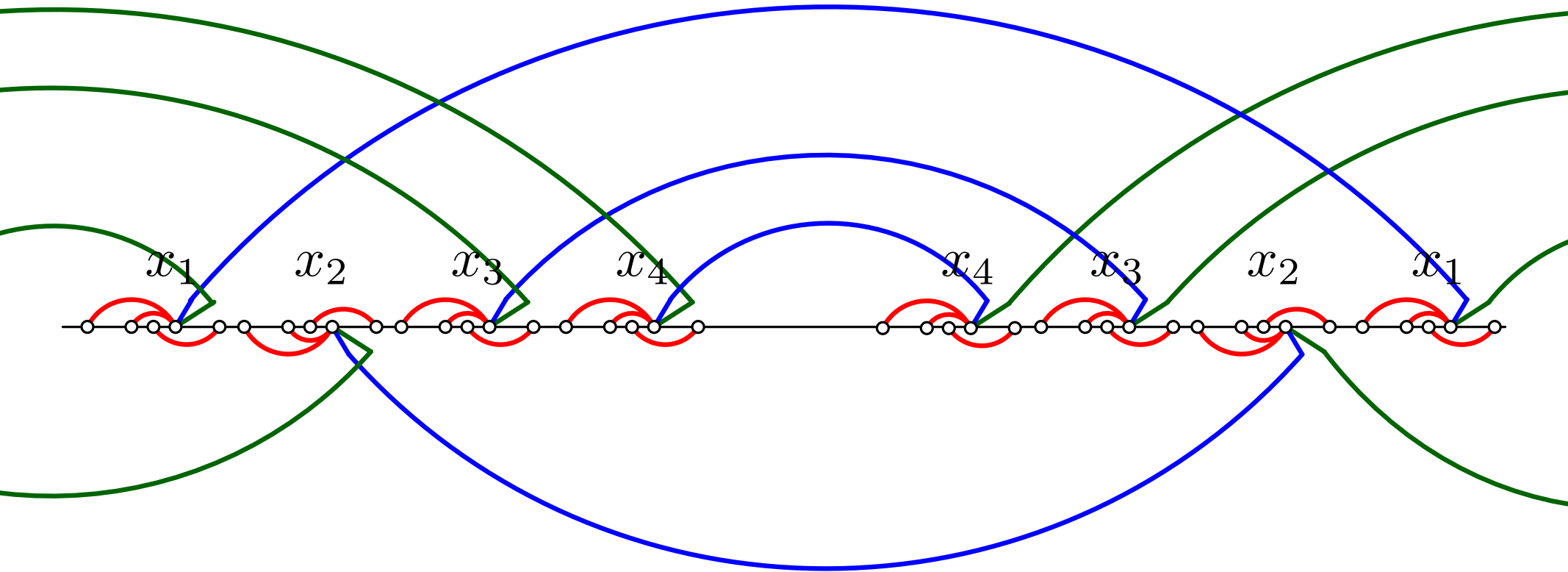
Reduction also works for two colors:

- ▶ subdivide some edges and use different colors

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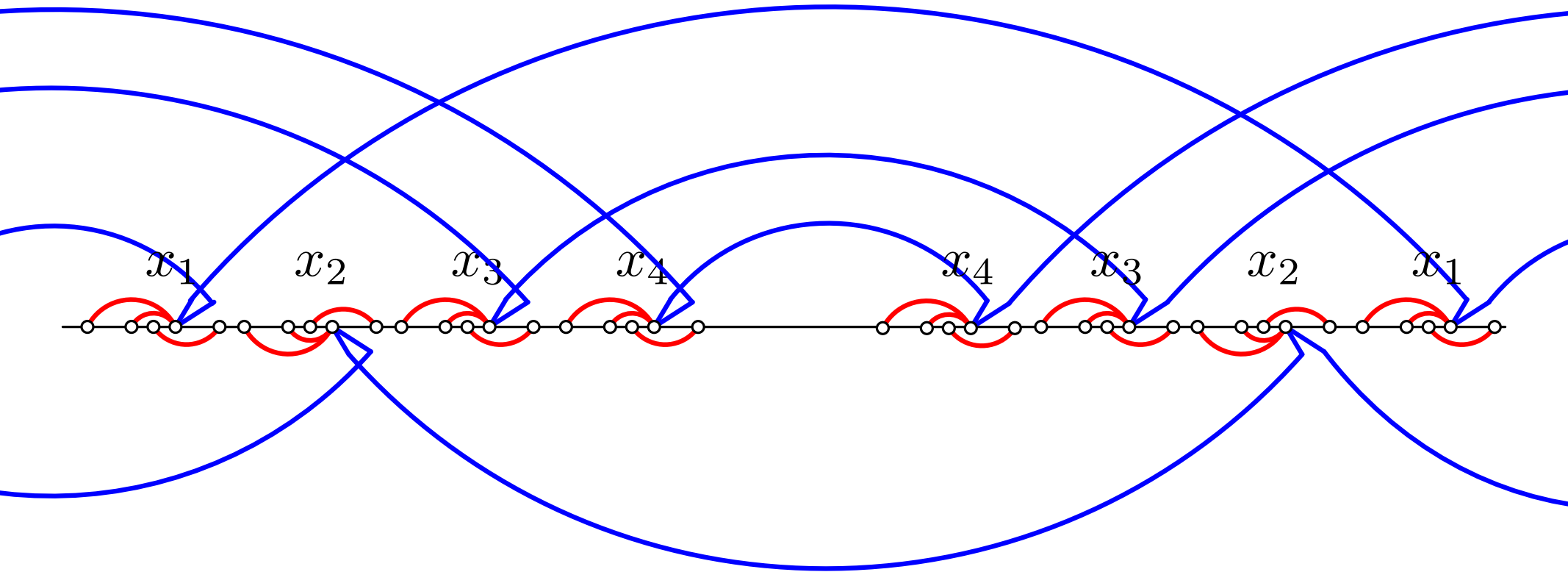
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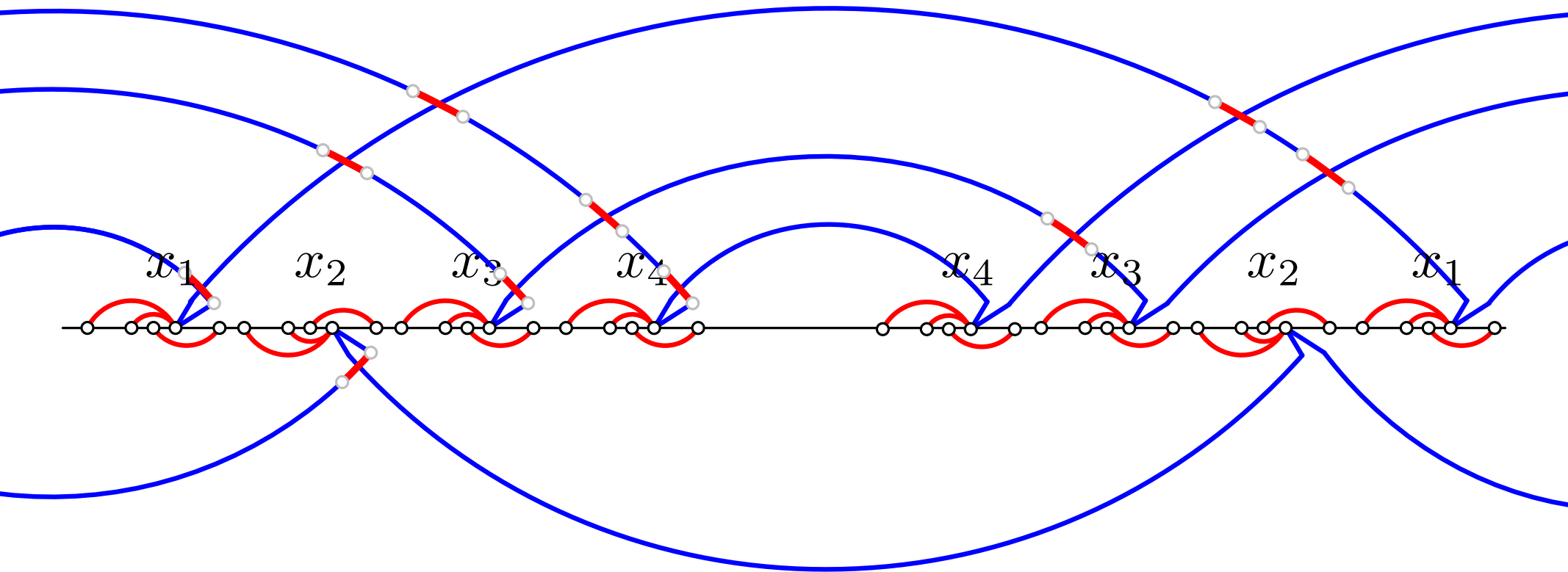
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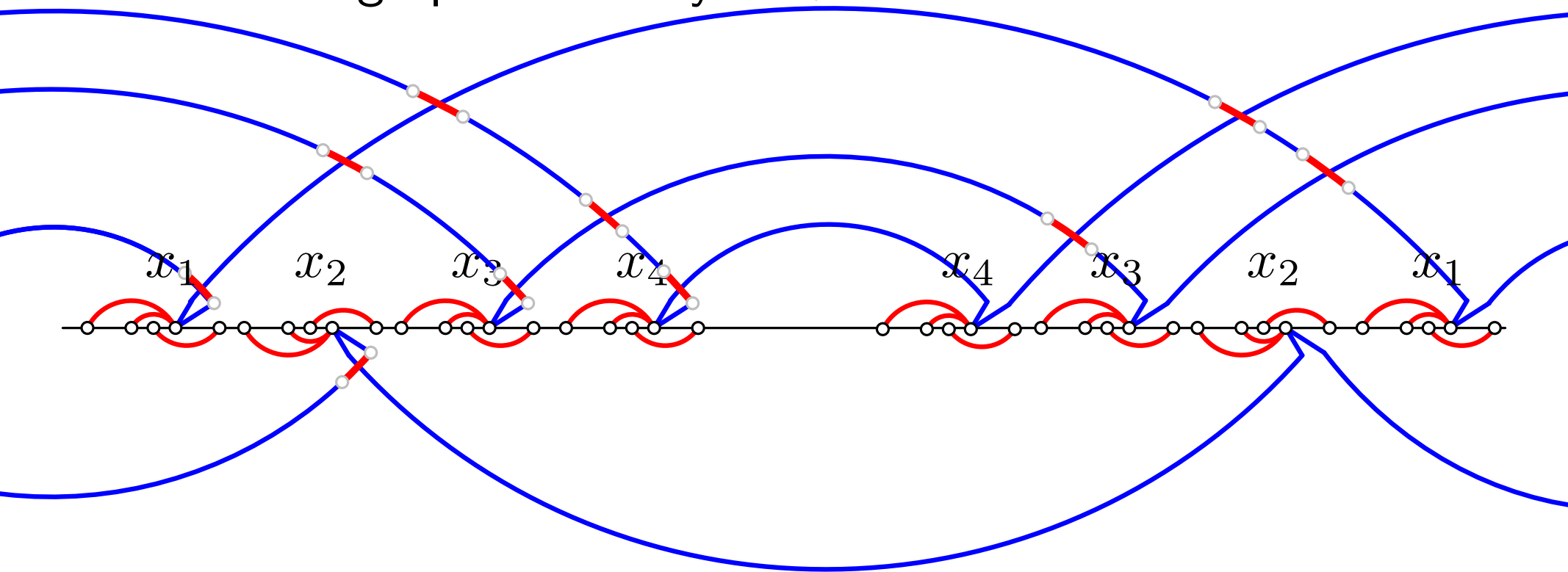
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Reduction also works for two colors:

- ▶ subdivide some edges and use different colors
- ▶ common graph now is cycle + isolated vertices

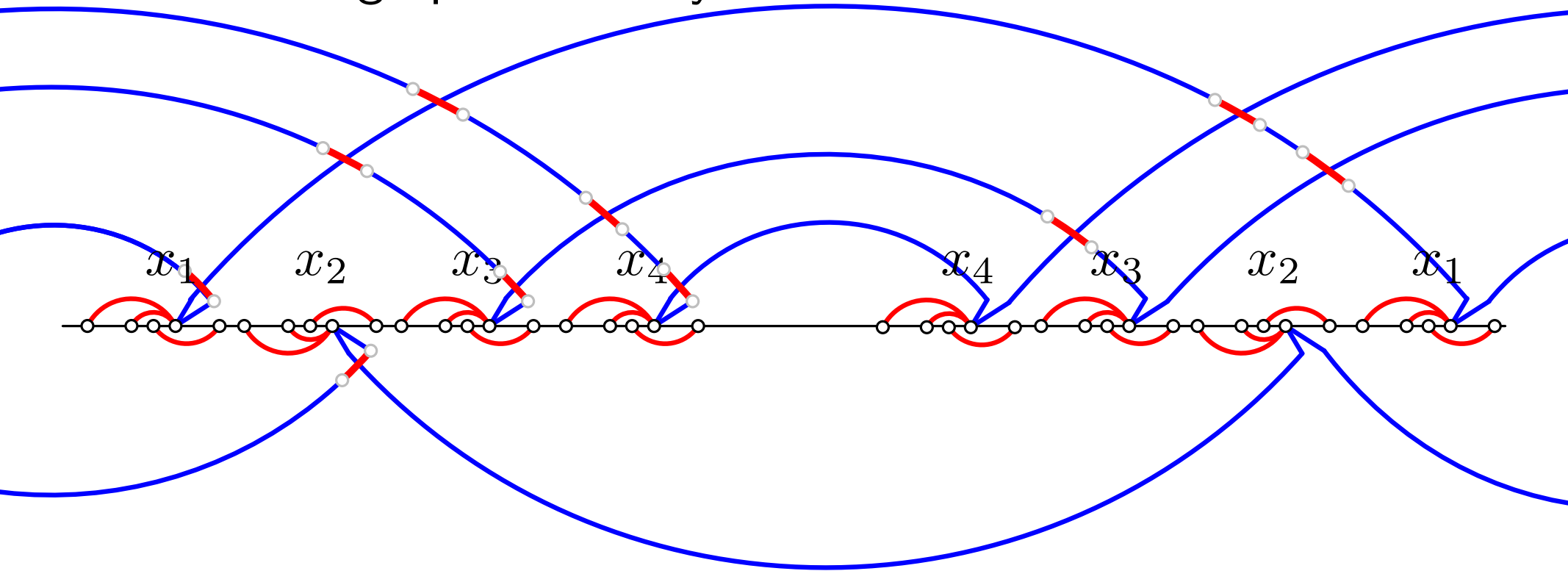




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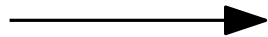


## Theorem.

It is NP-complete to decide whether two graphs  $G_1, G_2$  whose common graph consists of a cycle plus isolated vertices admit an ORTHOSEFE.

# Our Results

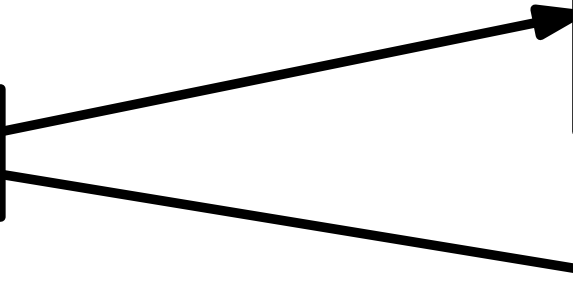
reduces to



NAE-3SAT

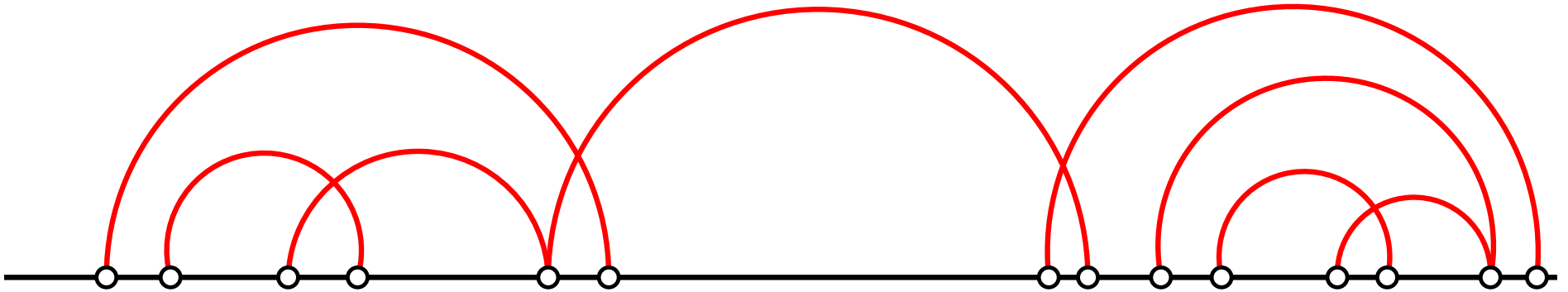
$G$  cycle, 3 colors  
two colors maxdeg-3 + outerplanar

$G$  cycle + isolated  
vertices, 2 colors



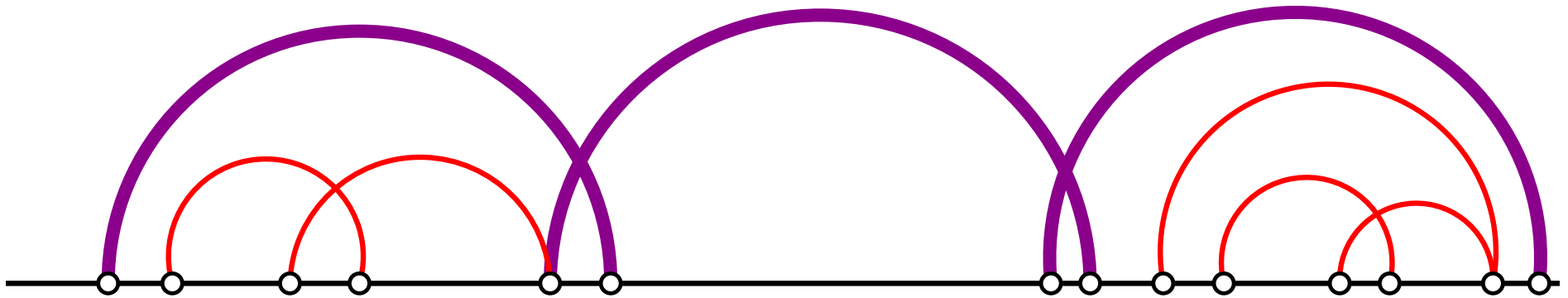
$G$  cycle;  $G_1$  outerplanar,  $\deg \leq 3$

- ▶ Consider  $G_1 \cap G_2$  on a line and  $G_1$  above.
  - Nested intersection components
  - Bipartition of intersecting edges



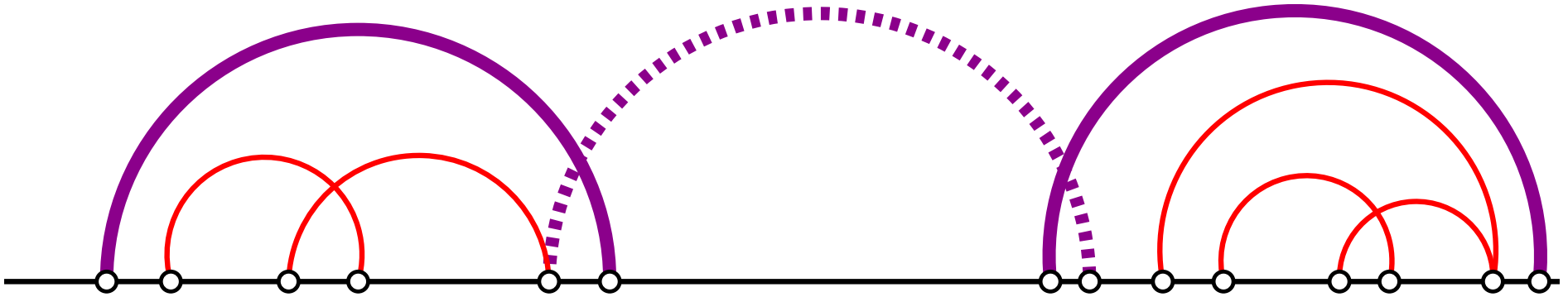
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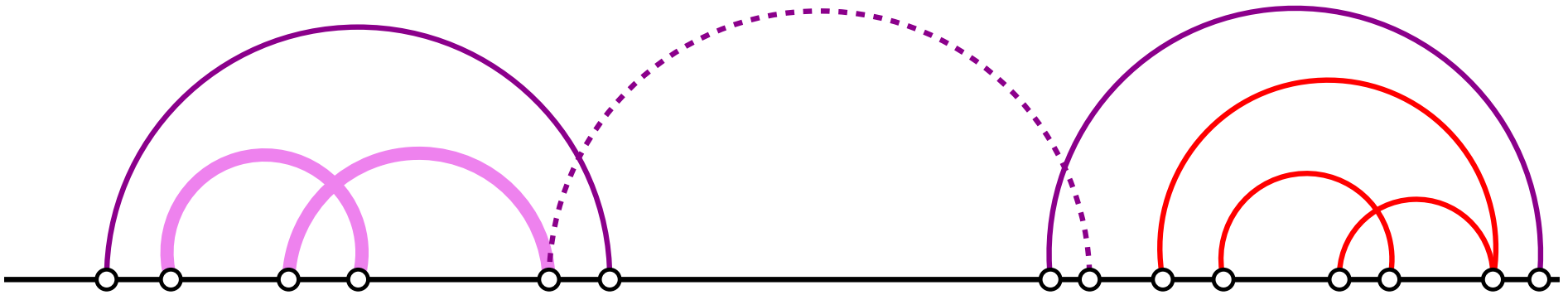
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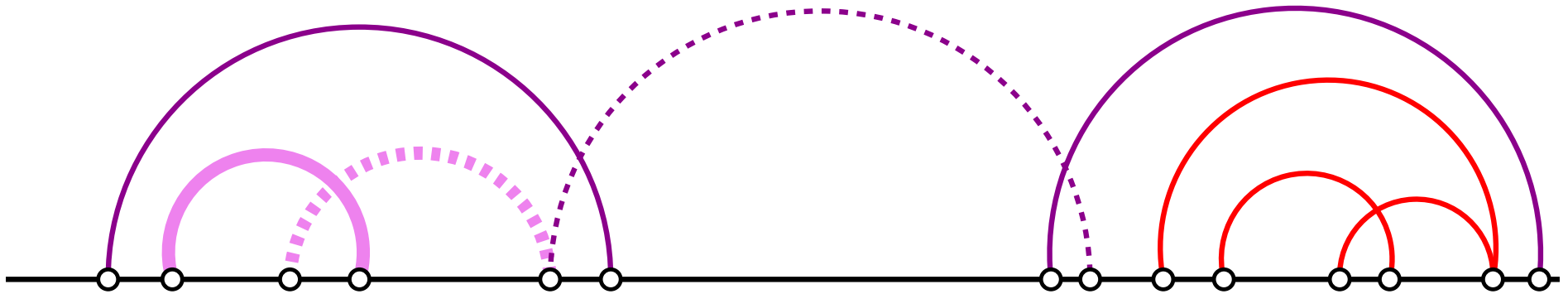
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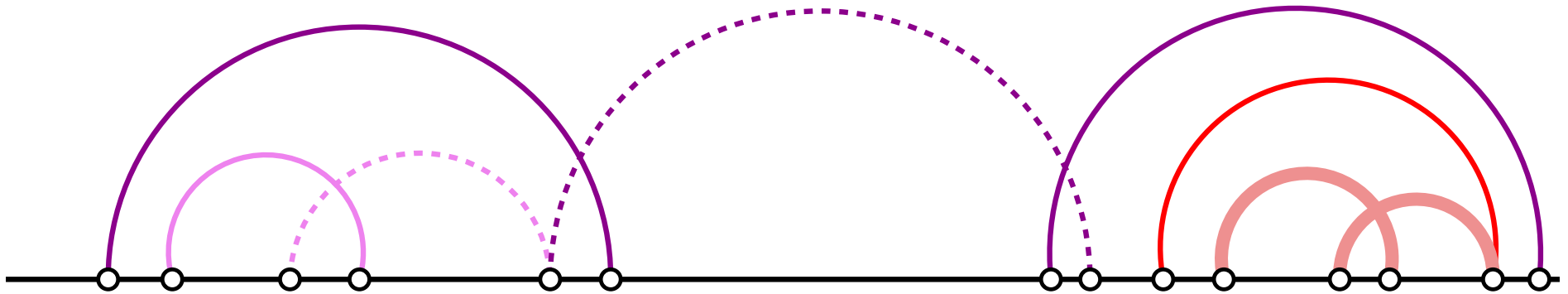
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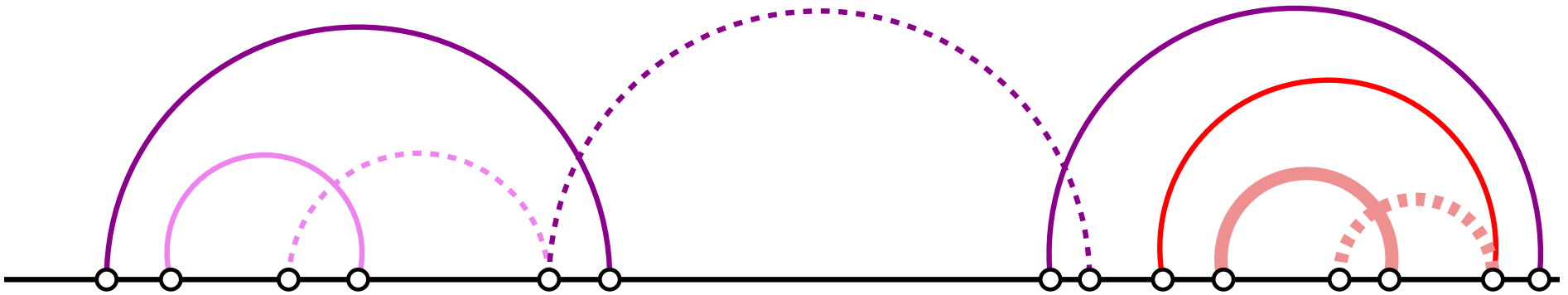
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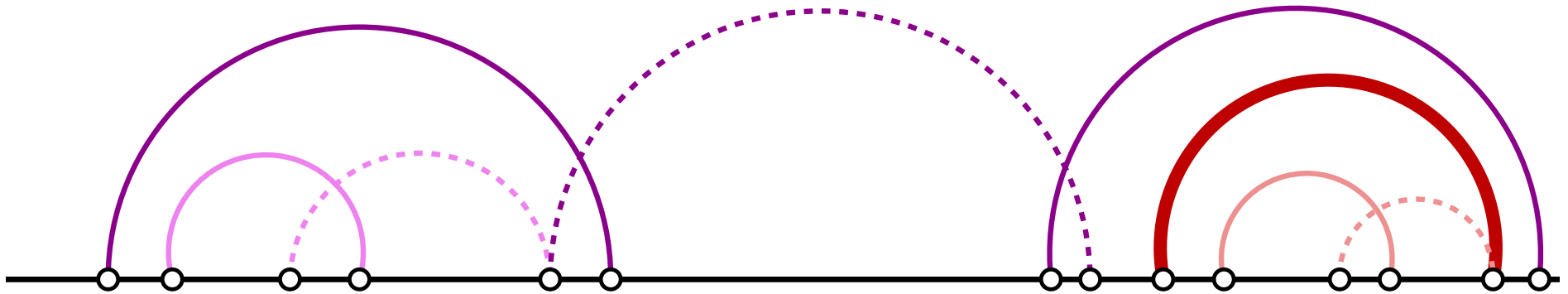
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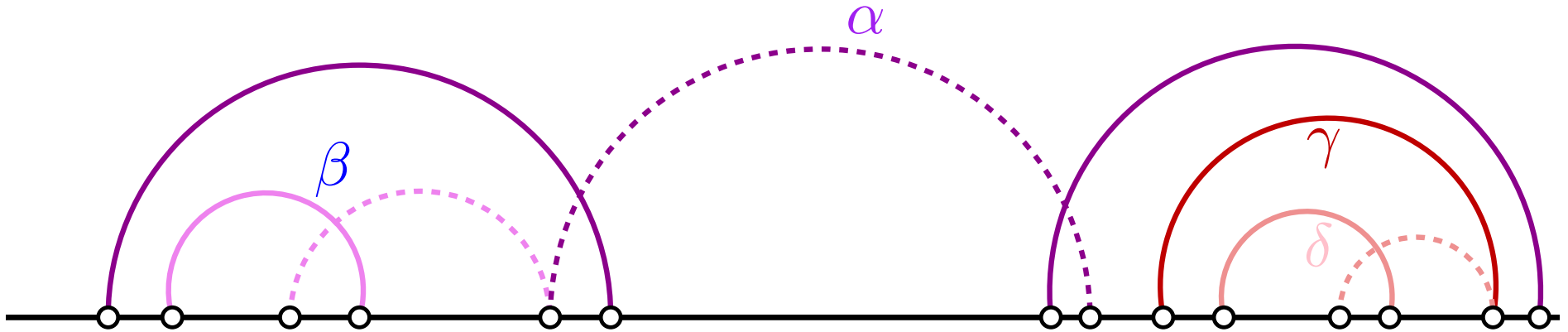
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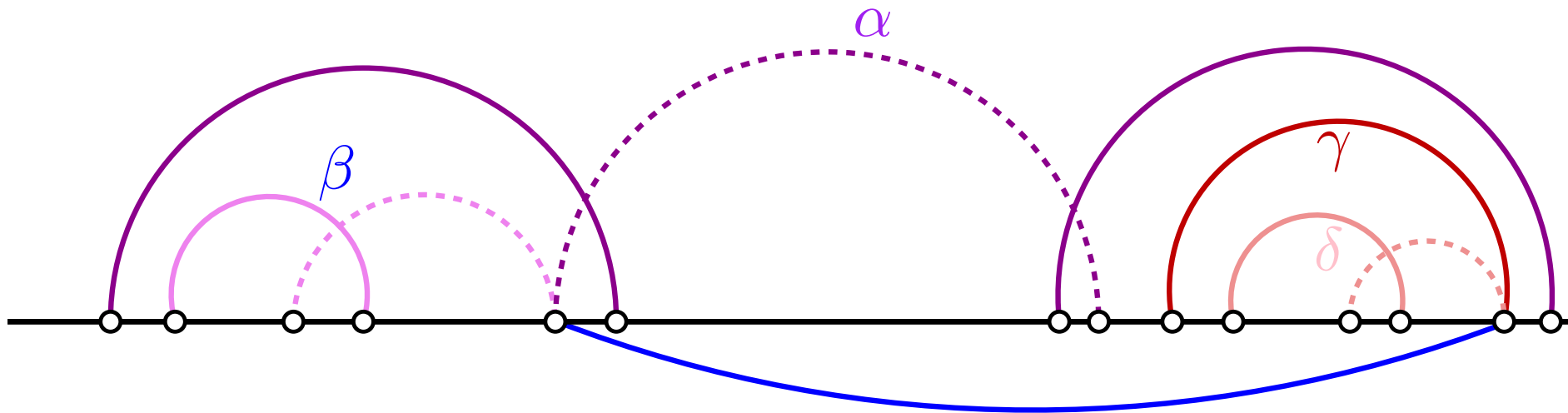
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- ▶ Boolean variable per class: dashed up = false

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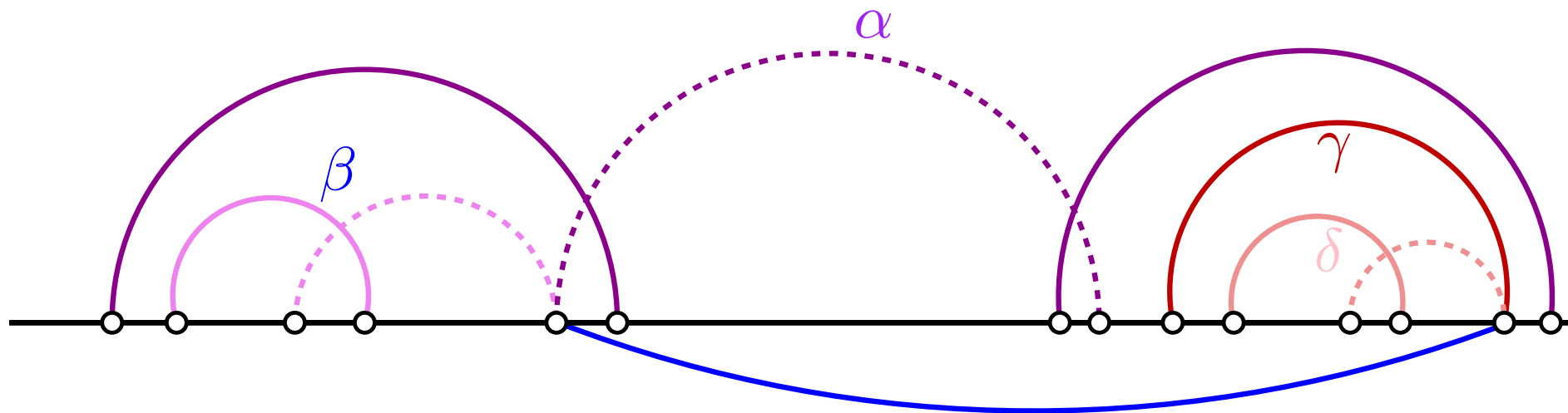
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$$\neg((\bar{\beta} \wedge \bar{\alpha} \wedge \bar{\gamma} \wedge \delta) \vee (\beta \wedge \alpha \wedge \gamma \wedge \bar{\delta}))$$

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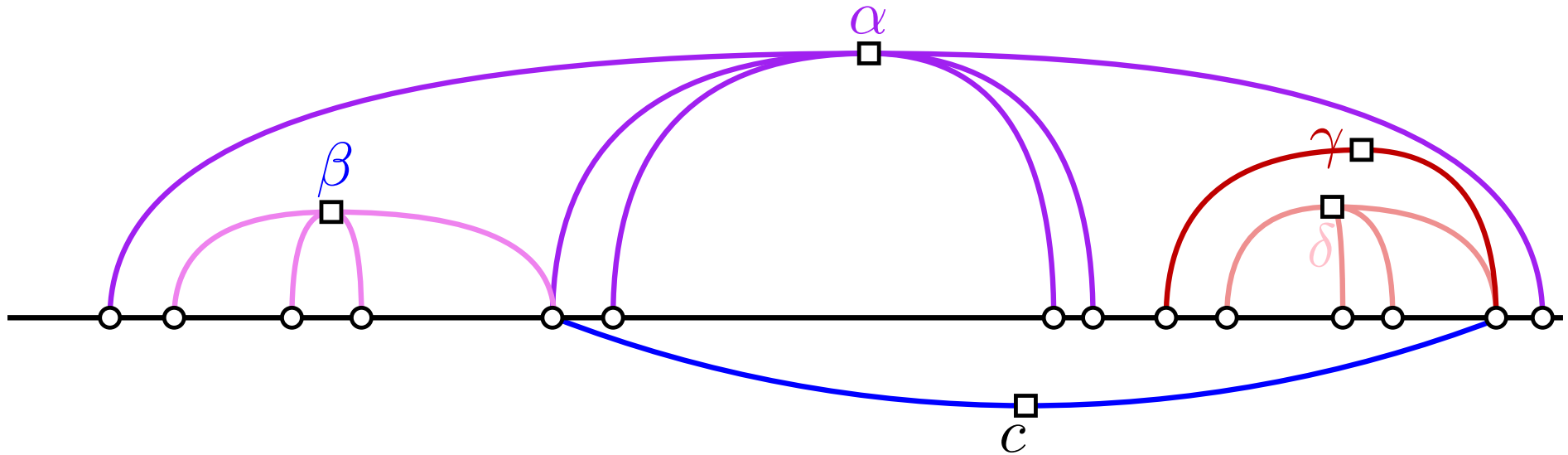
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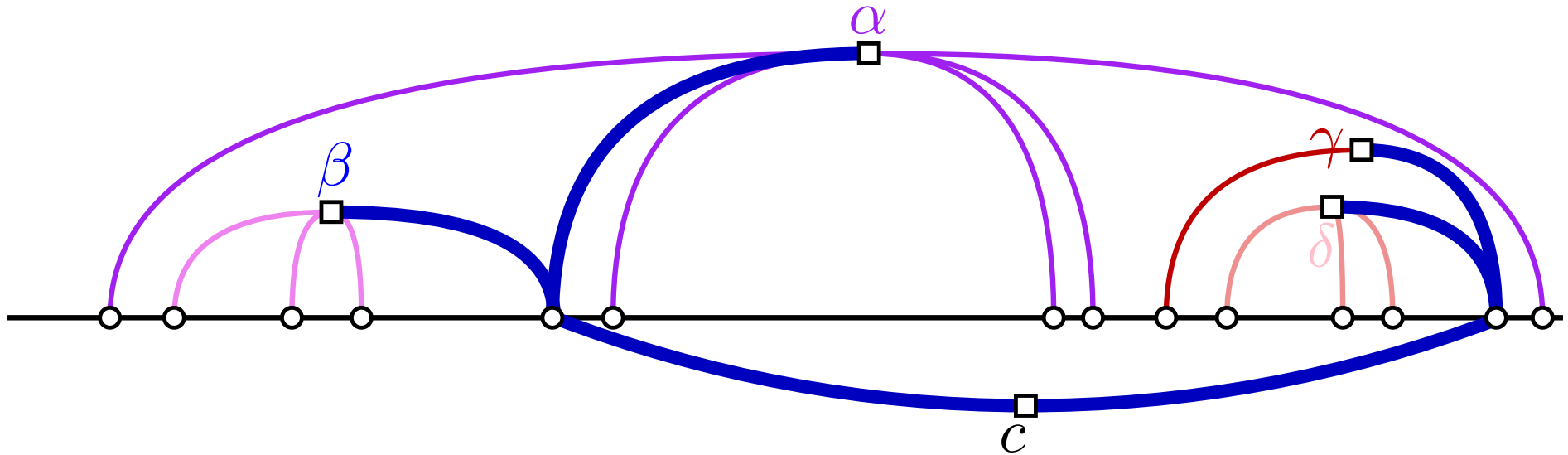
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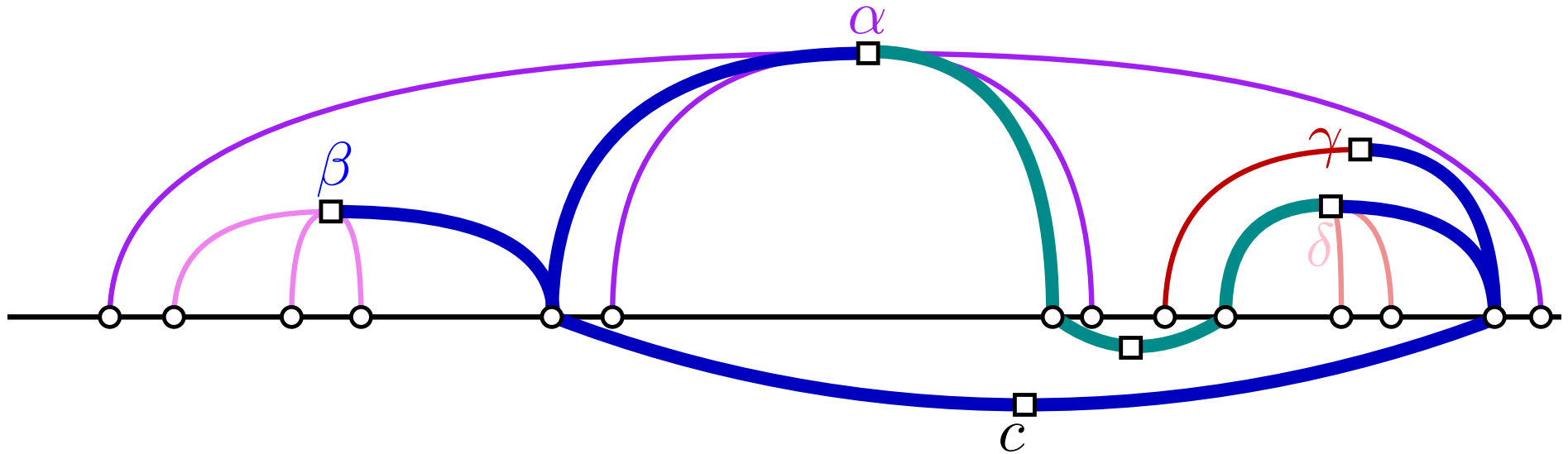
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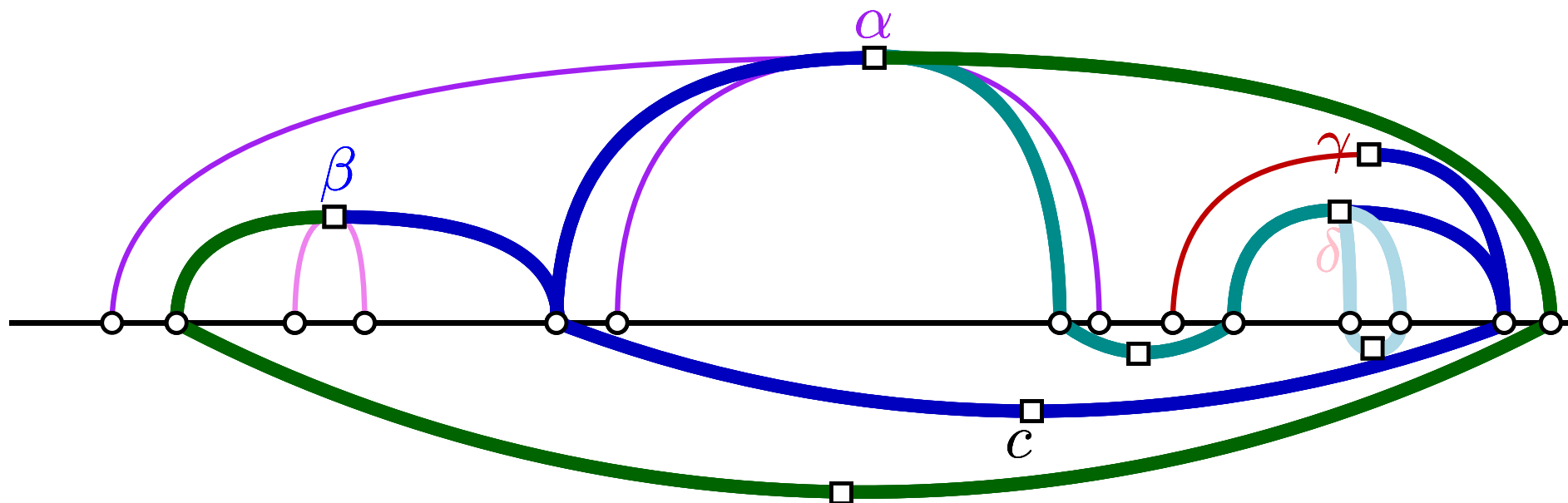


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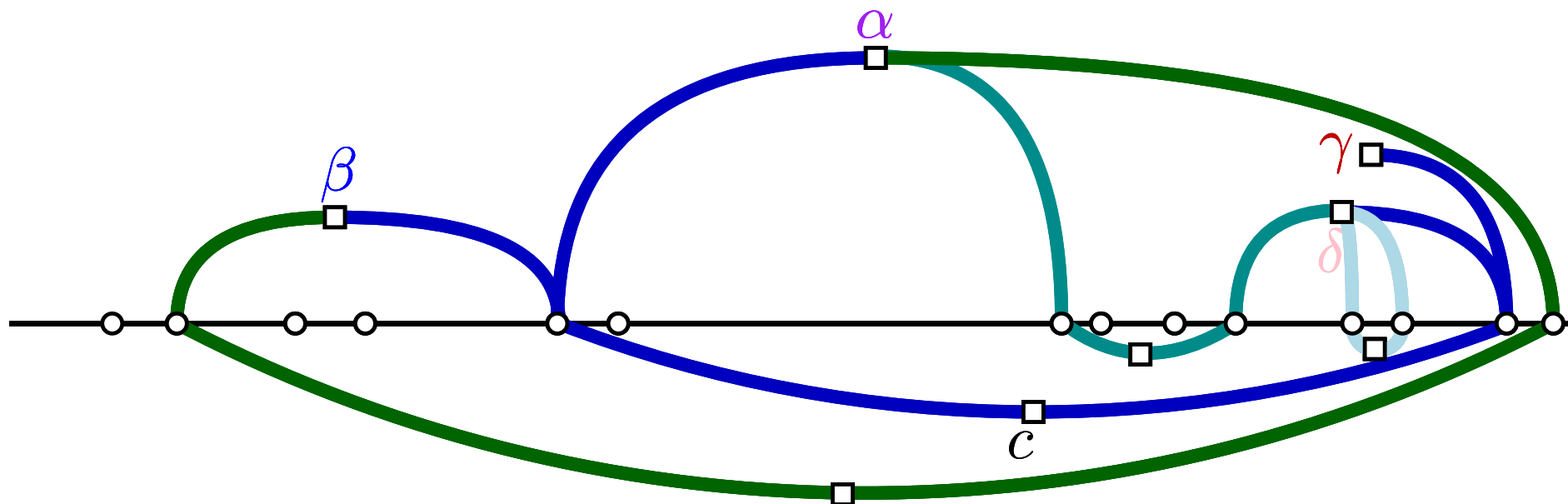
- ▶ Consider  $G_1 \cap G_2$  on a line and  $G_1$  above.
  - Nested intersection components
  - Bipartition of intersecting edges



- ▶ Boolean variable per class: dashed up = false
- ▶ Blue can be inserted iff not one end vertex up, one down  
$$\neg((\bar{\beta} \wedge \bar{\alpha} \wedge \bar{\gamma} \wedge \delta) \vee (\beta \wedge \alpha \wedge \gamma \wedge \bar{\delta}))$$
- ▶ not-all-equal SAT

## $G$ cycle; $G_1$ outerplanar, $\text{deg} \leq 3$

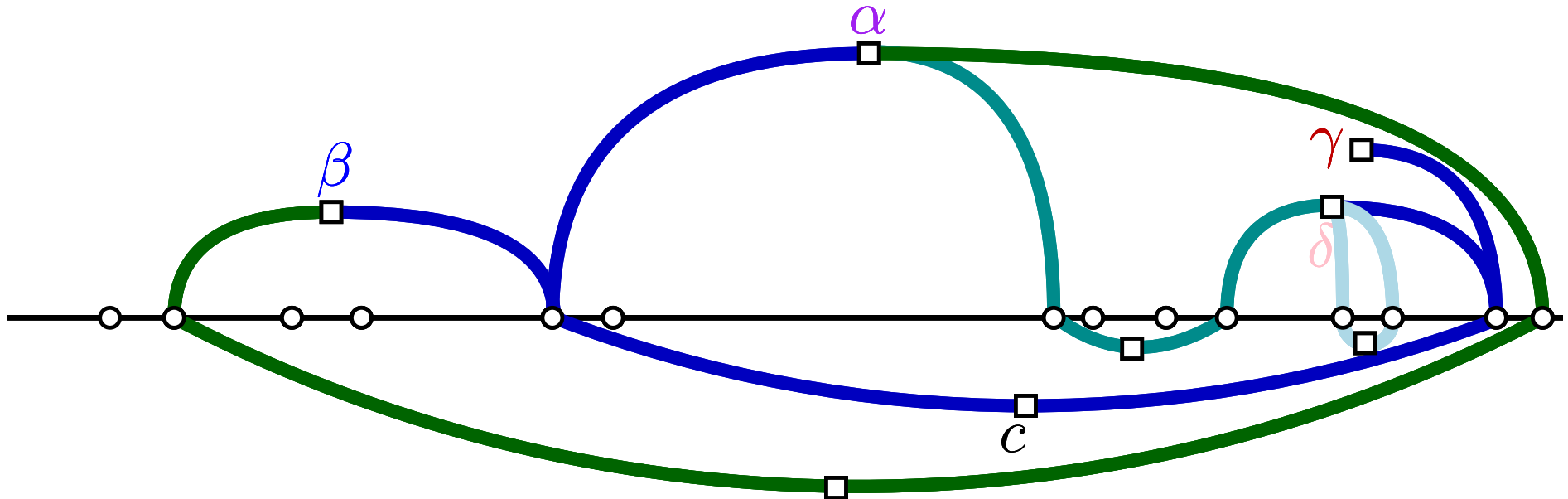
- ▶ Consider  $G_1 \cap G_2$  on a line and  $G_1$  above.
  - Nested intersection components
  - Bipartition of intersecting edges



- ▶ Boolean variable per class: dashed up = false
- ▶ Blue can be inserted iff not one end vertex up, one down  
$$\neg((\bar{\beta} \wedge \bar{\alpha} \wedge \bar{\gamma} \wedge \delta) \vee (\beta \wedge \alpha \wedge \gamma \wedge \bar{\delta}))$$
- ▶ planar not-all-equal SAT

## $G$ cycle; $G_1$ outerplanar, $\text{deg} \leq 3$

- ▶ Consider  $G_1 \cap G_2$  on a line and  $G_1$  above.
  - Nested intersection components
  - Bipartition of intersecting edges



- ▶ Boolean variable per class: dashed up = false
- ▶ Blue can be inserted iff not one end vertex up, one down  
$$\neg((\bar{\beta} \wedge \bar{\alpha} \wedge \bar{\gamma} \wedge \delta) \vee (\beta \wedge \alpha \wedge \gamma \wedge \bar{\delta}))$$
- ▶ planar not-all-equal SAT, which is in  $\mathcal{P}$ ! [Moret '88]

# Our Results

reduces to



NAE-3SAT

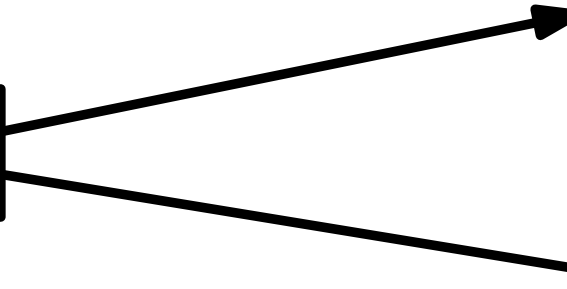
$G$  cycle, 3 colors  
two colors maxdeg-3 + outerplanar

$G$  cycle + isolated  
vertices, 2 colors

$G$  cycle

$G_1$  maxdeg-3  
+ outerplanar

Planar NAE-SAT



# Our Results

reduces to



NAE-3SAT

$G$  cycle, 3 colors  
two colors maxdeg-3 + outerplanar

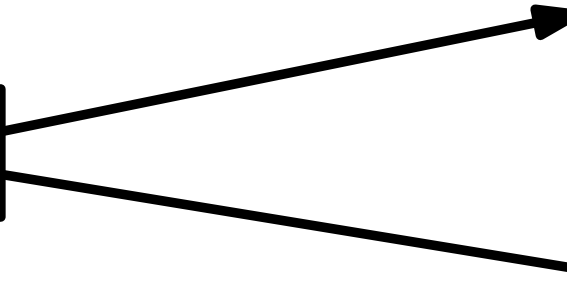
$G$  cycle + isolated vertices, 2 colors

$G$  cycle

$G_1$  maxdeg-3

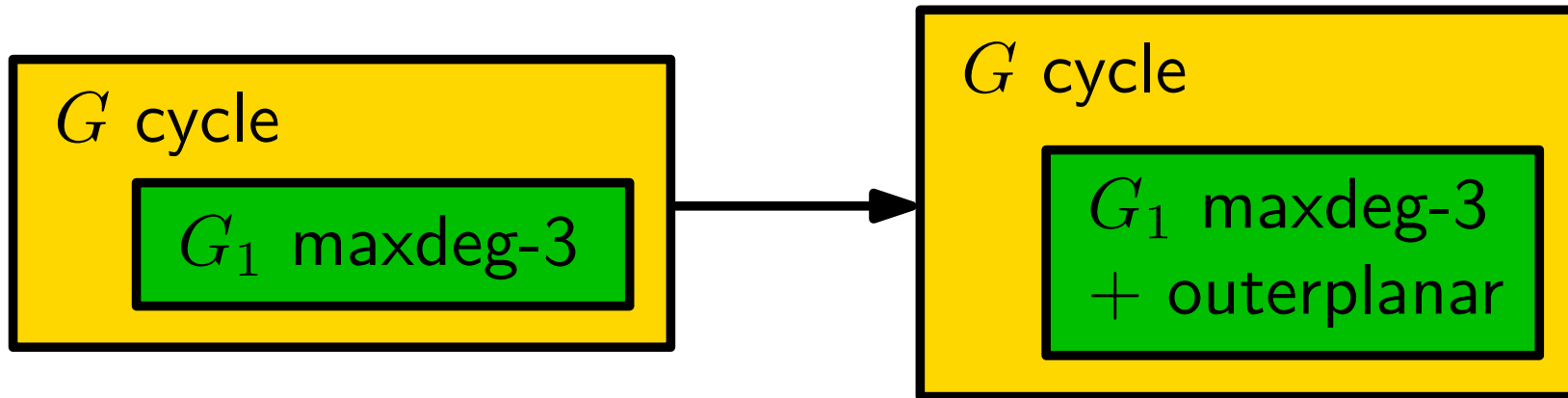
$G_1$  maxdeg-3 + outerplanar

Planar NAE-SAT



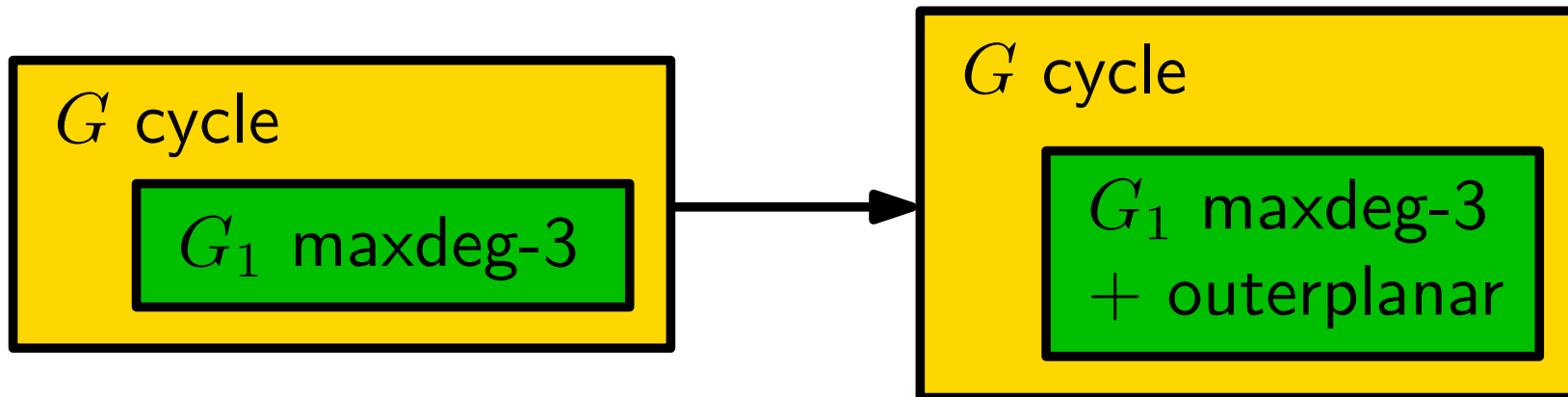
# Making a Maxdeg-3 Graph Outerplanar

**Theorem.**

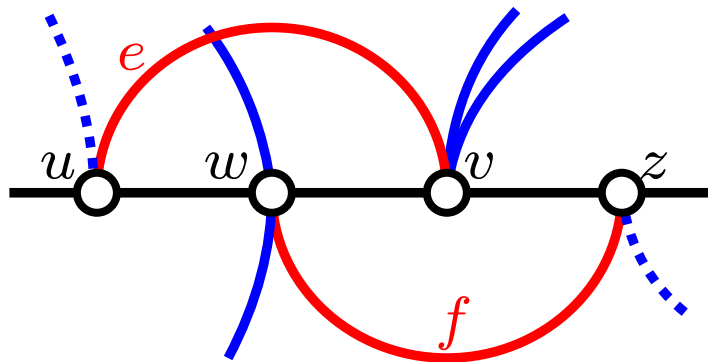


# Making a Maxdeg-3 Graph Outerplanar

**Theorem.**

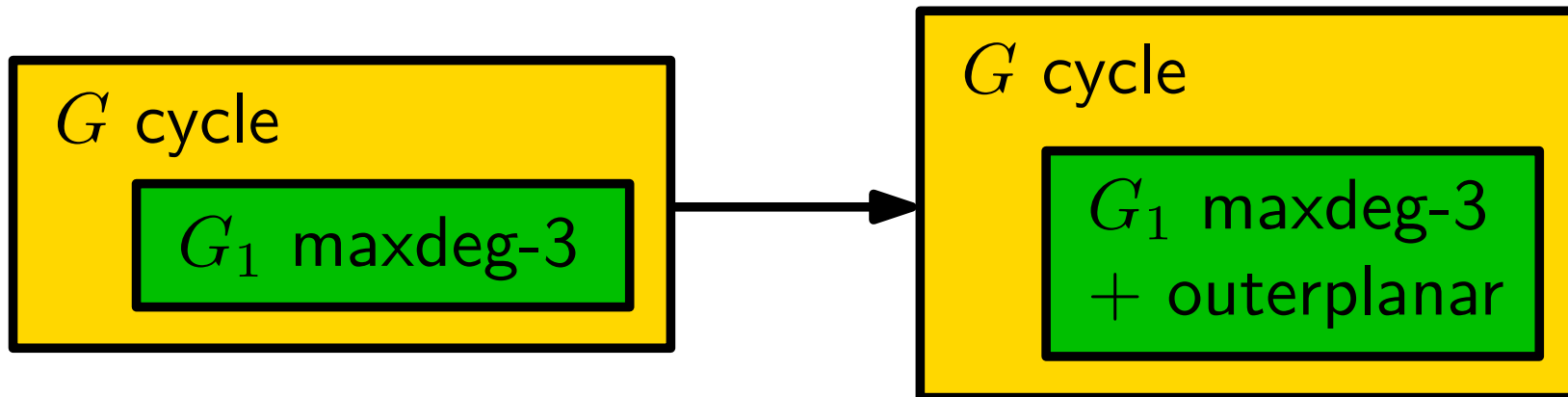


Proof:

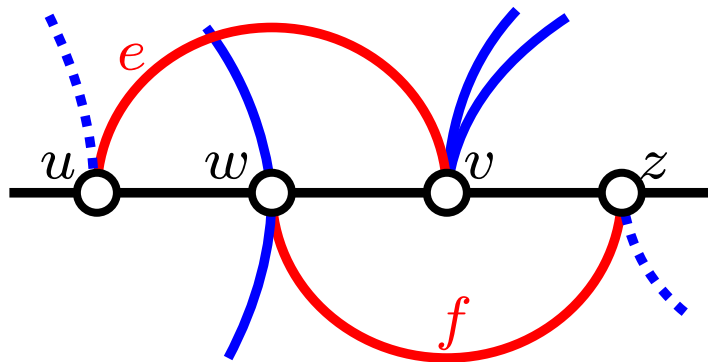


# Making a Maxdeg-3 Graph Outerplanar

**Theorem.**



Proof:

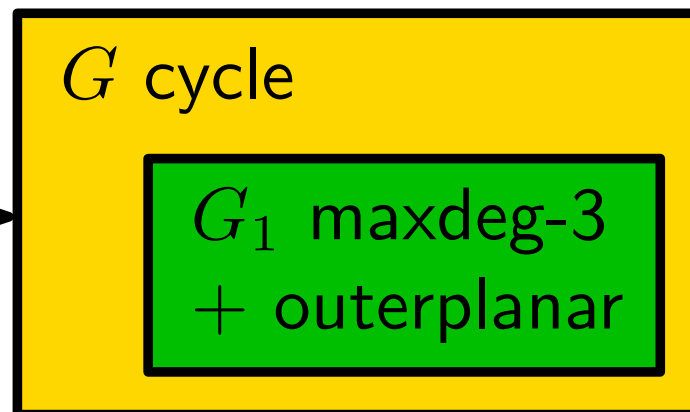
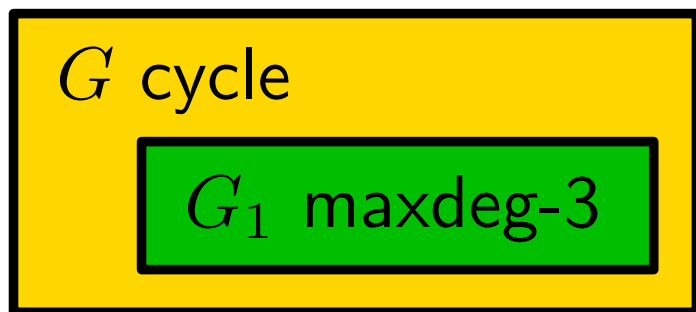


pick  $u, z$  as close as possible

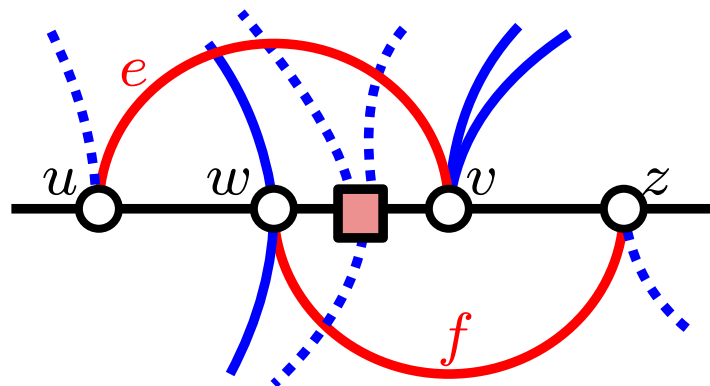


# Making a Maxdeg-3 Graph Outerplanar

**Theorem.**



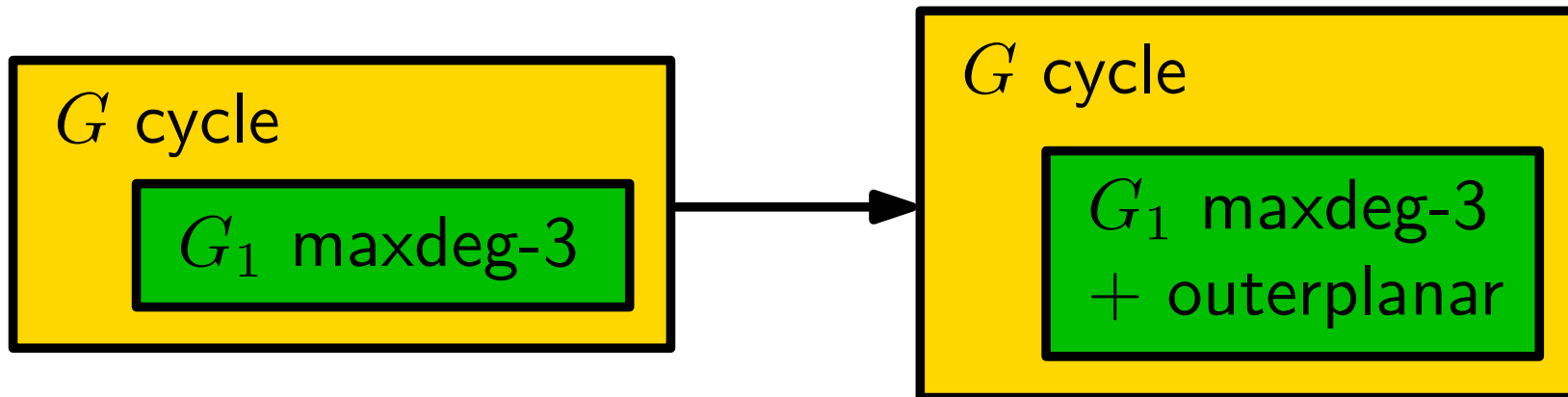
Proof:



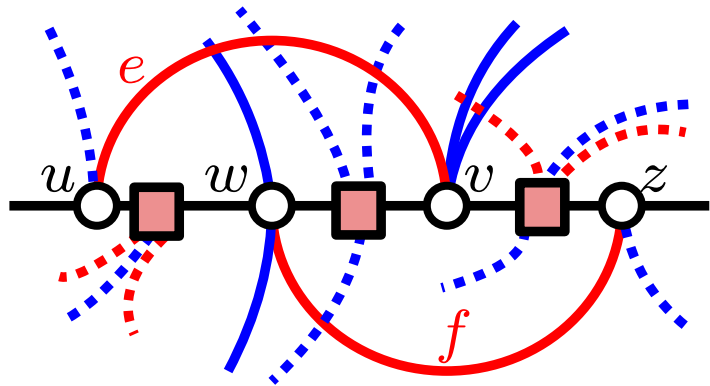
pick  $u, z$  as close as possible

# Making a Maxdeg-3 Graph Outerplanar

**Theorem.**



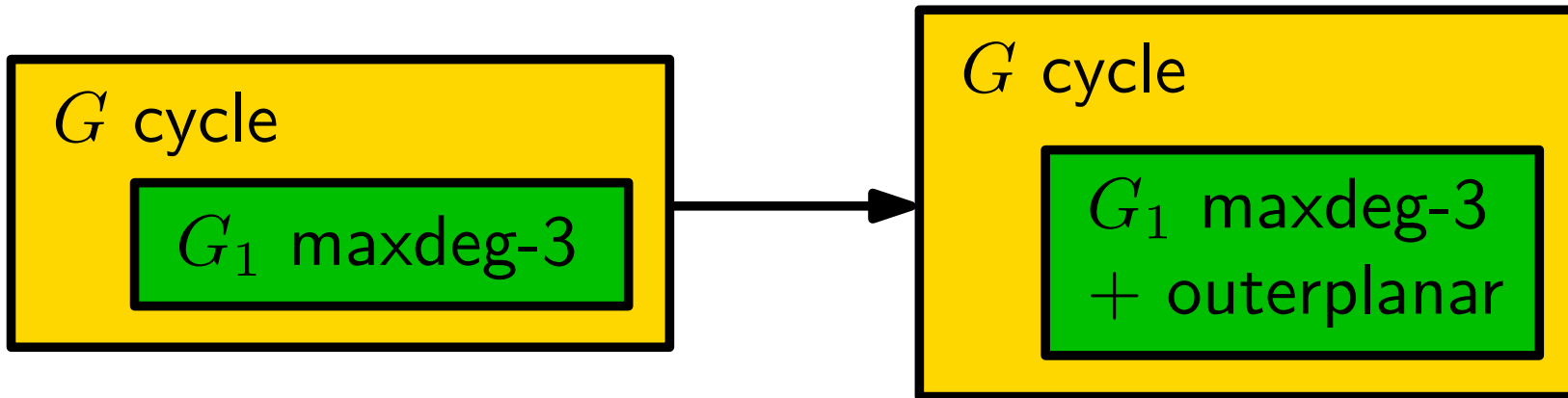
Proof:



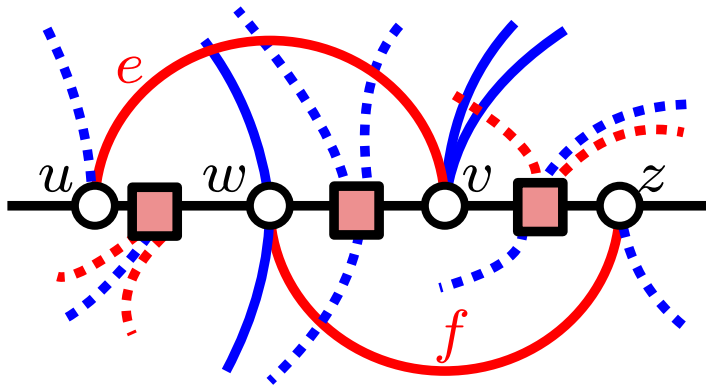
pick  $u, z$  as close as possible

# Making a Maxdeg-3 Graph Outerplanar

**Theorem.**



Proof:

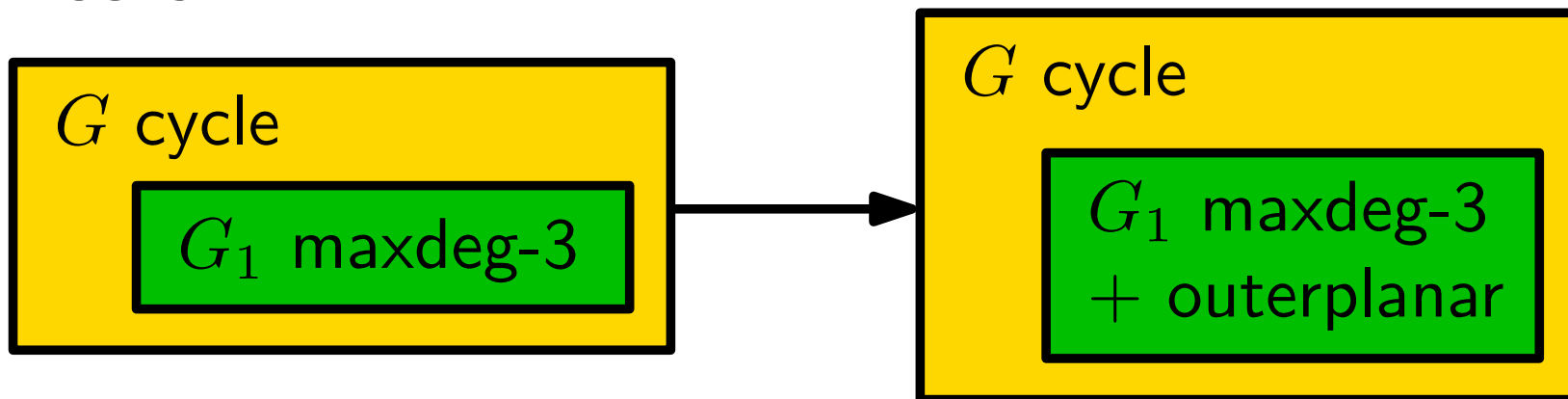


pick  $u, z$  as close as possible

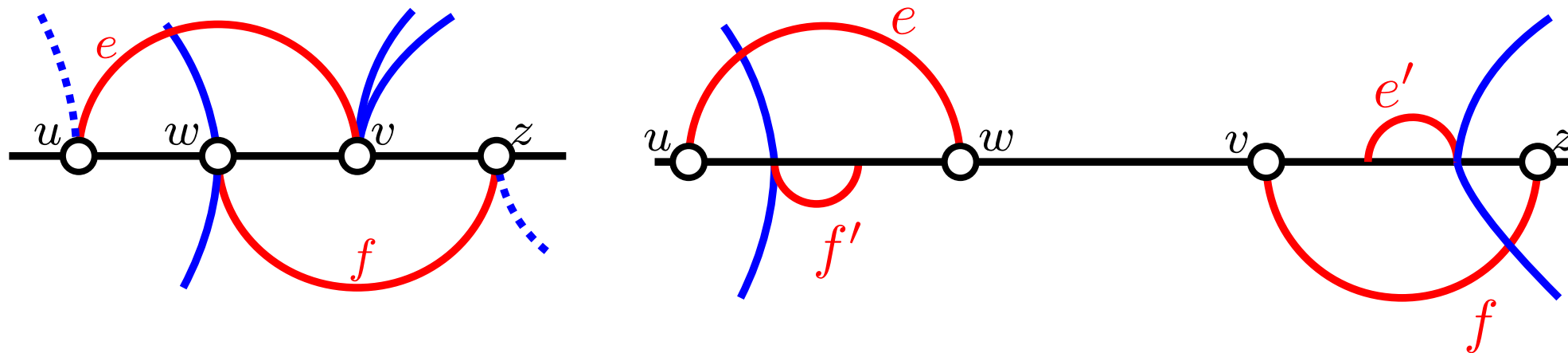
- ▶  $\blacksquare$  outerplanar,
- ▶ no edge between  $\blacksquare$  and  $u, z$

# Making a Maxdeg-3 Graph Outerplanar

**Theorem.**

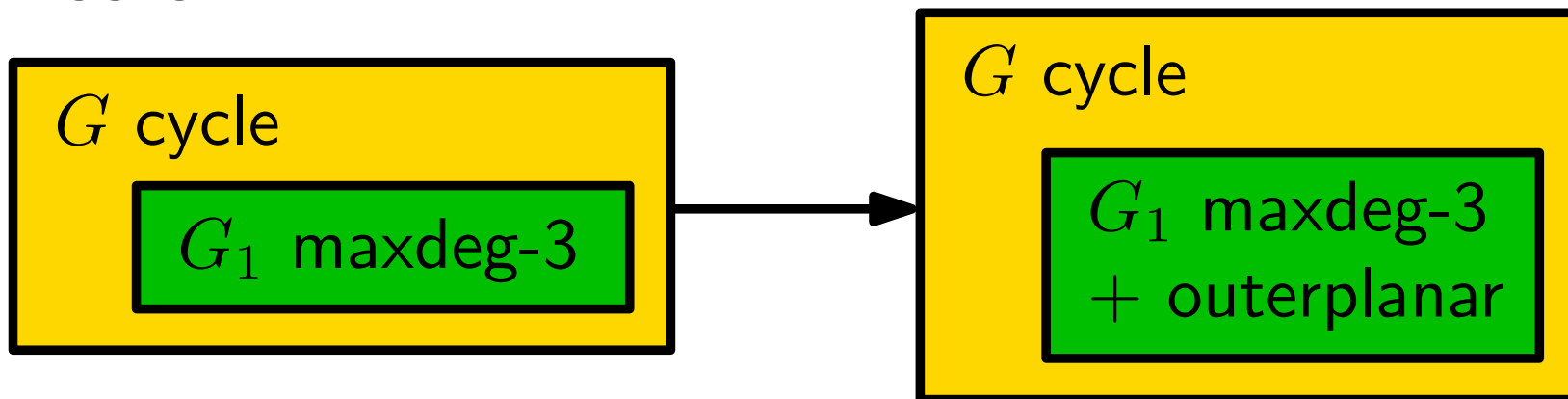


Proof:

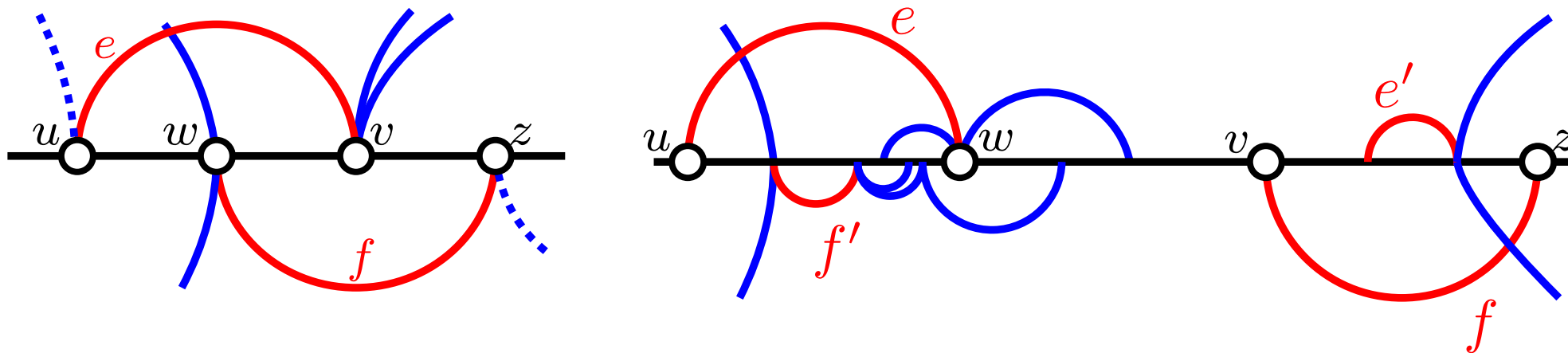


# Making a Maxdeg-3 Graph Outerplanar

**Theorem.**

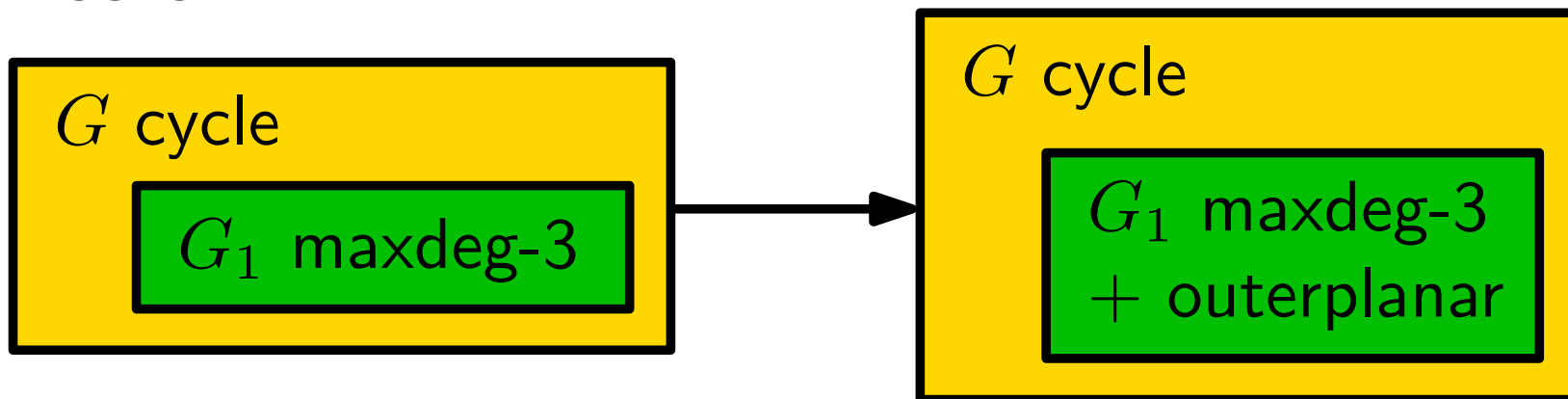


Proof:

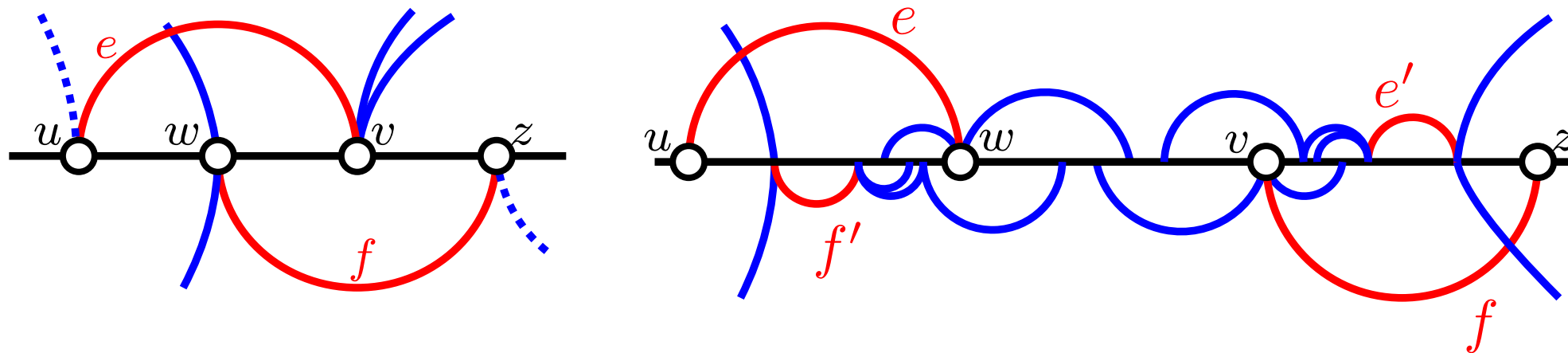


# Making a Maxdeg-3 Graph Outerplanar

**Theorem.**

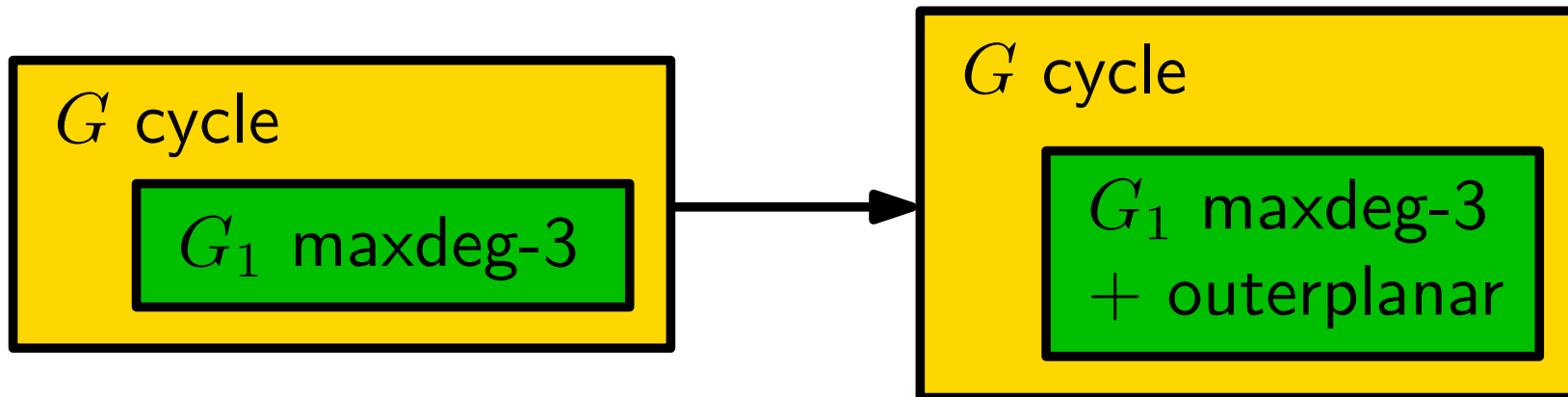


Proof:

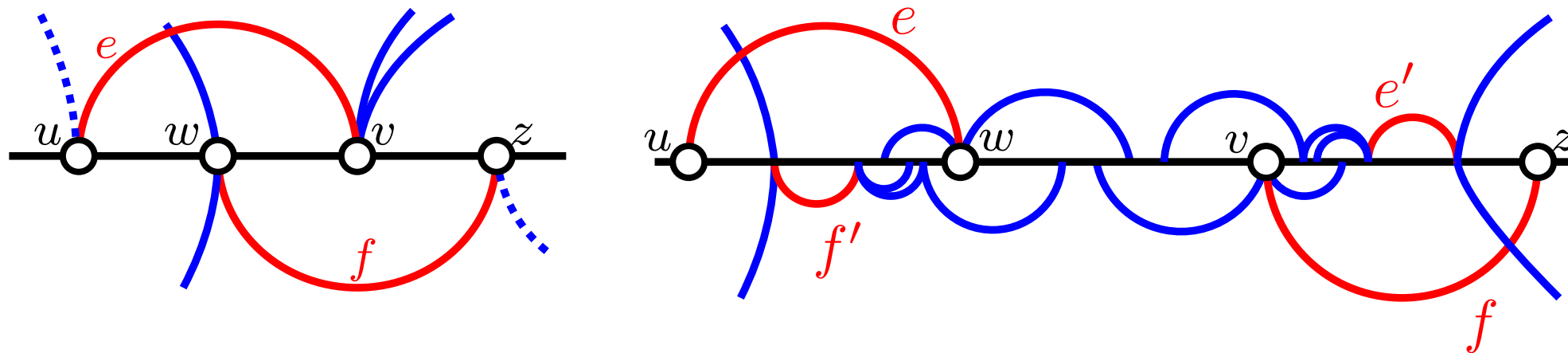


# Making a Maxdeg-3 Graph Outerplanar

**Theorem.**



Proof:



# Our Results

reduces to



NAE-3SAT

$G$  cycle, 3 colors  
two colors maxdeg-3 + outerplanar

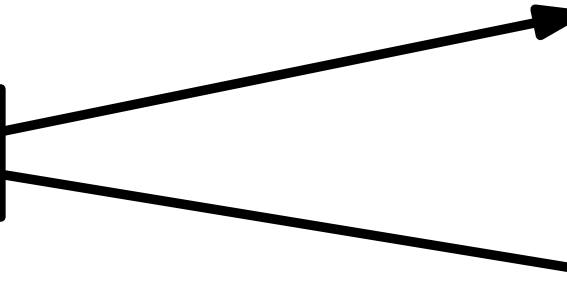
$G$  cycle + isolated  
vertices, 2 colors

$G$  cycle

$G_1$  maxdeg-3

$G_1$  maxdeg-3  
+ outerplanar

Planar NAE-SAT





# Our Results

reduces to



NAE-3SAT

$G$  cycle, 3 colors  
two colors maxdeg-3 + outerplanar

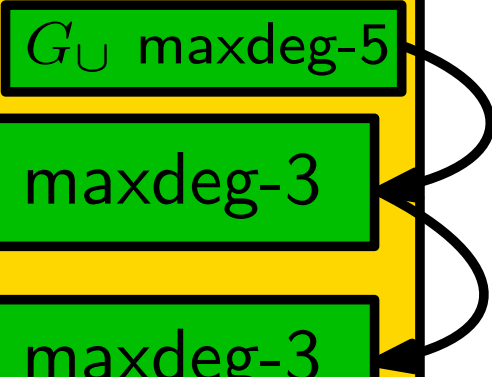
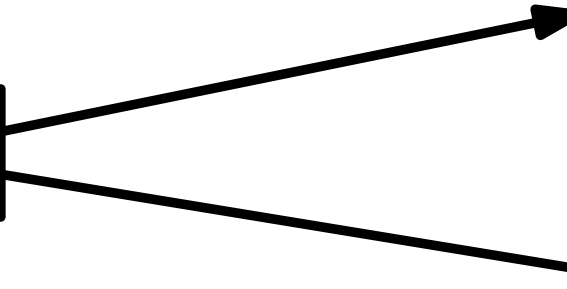
$G$  cycle + isolated vertices, 2 colors

$G$  cycle  $G_U$  maxdeg-5

$G_1$  maxdeg-3

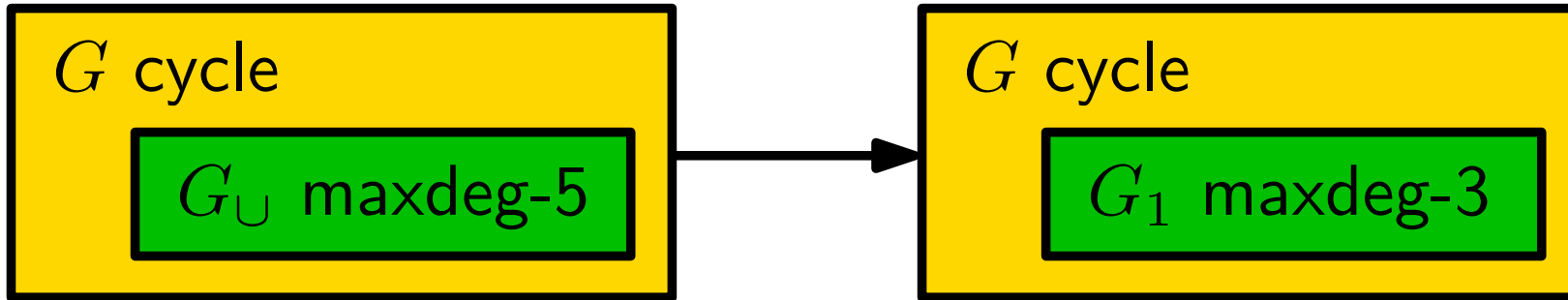
$G_1$  maxdeg-3 + outerplanar

Planar NAE-SAT



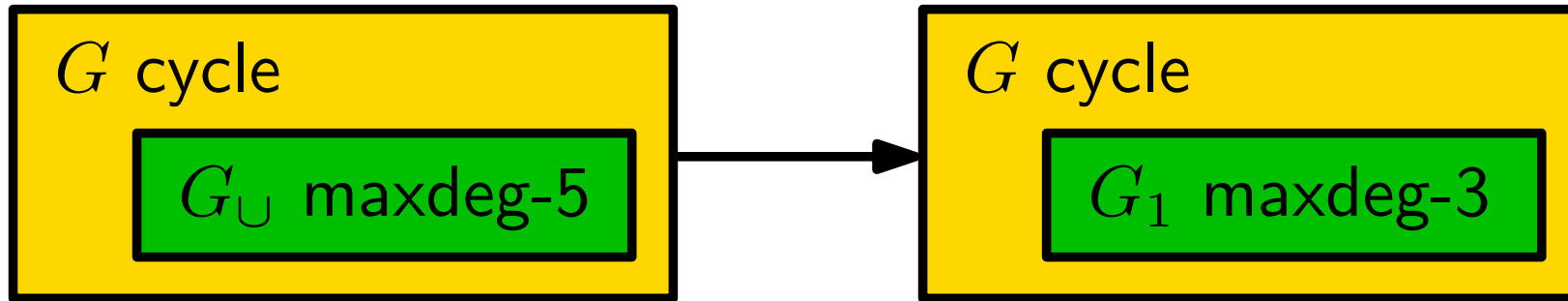
From  $G_n$  maxdeg-5 to  $G_1$  maxdeg-3

**Theorem.**

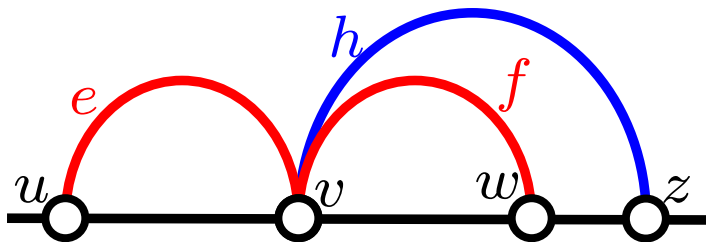


From  $G_n$  maxdeg-5 to  $G_1$  maxdeg-3

**Theorem.**

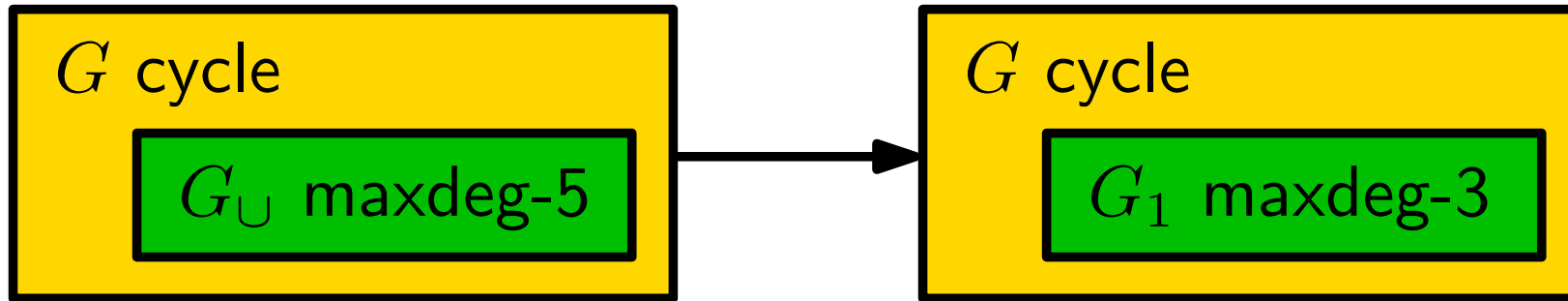


Proof:

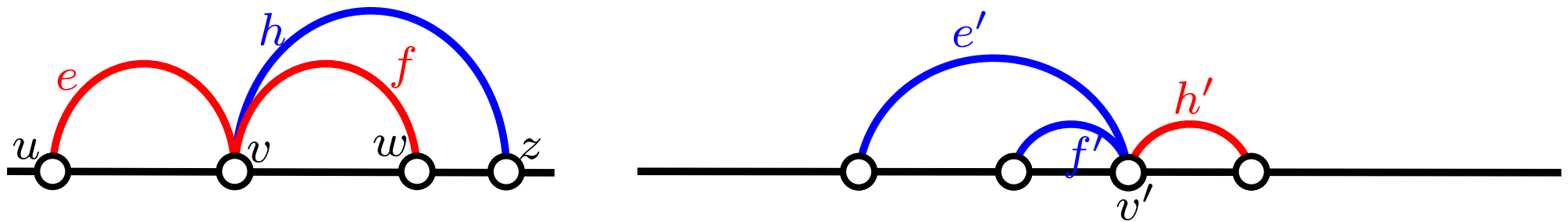


From  $G_n$  maxdeg-5 to  $G_1$  maxdeg-3

**Theorem.**

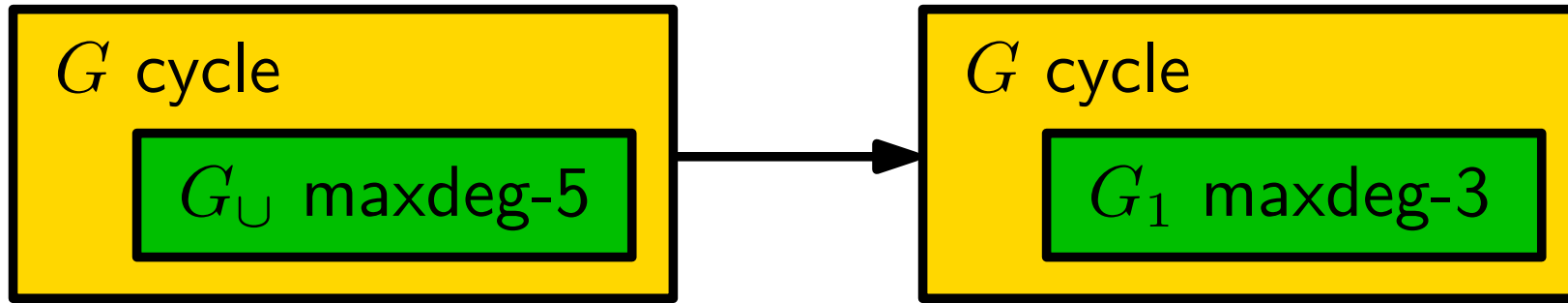


Proof:

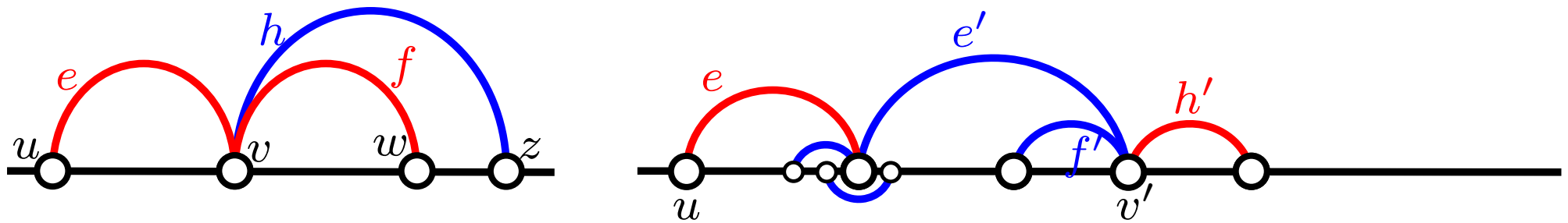


From  $G_n$  maxdeg-5 to  $G_1$  maxdeg-3

**Theorem.**

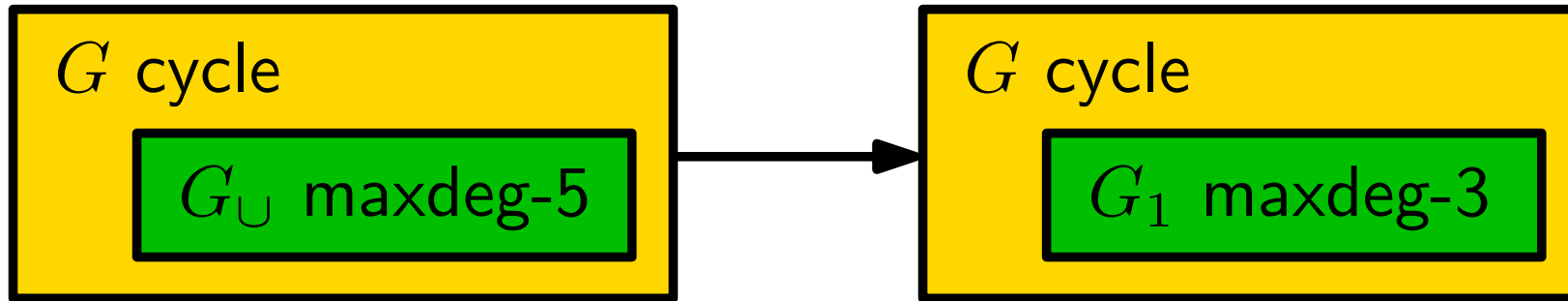


Proof:

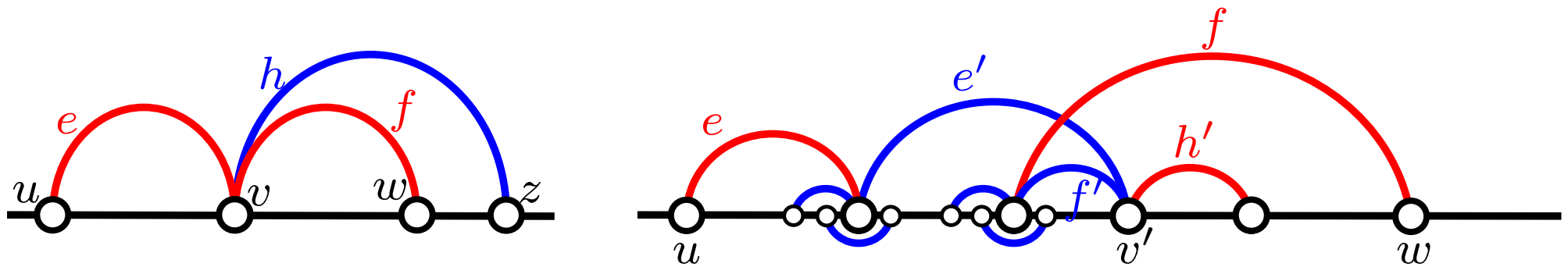


From  $G_n$  maxdeg-5 to  $G_1$  maxdeg-3

**Theorem.**

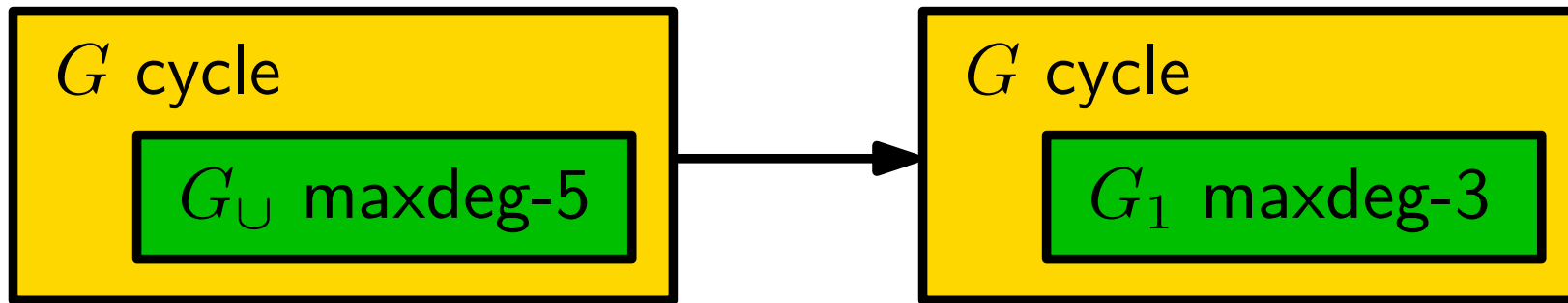


Proof:

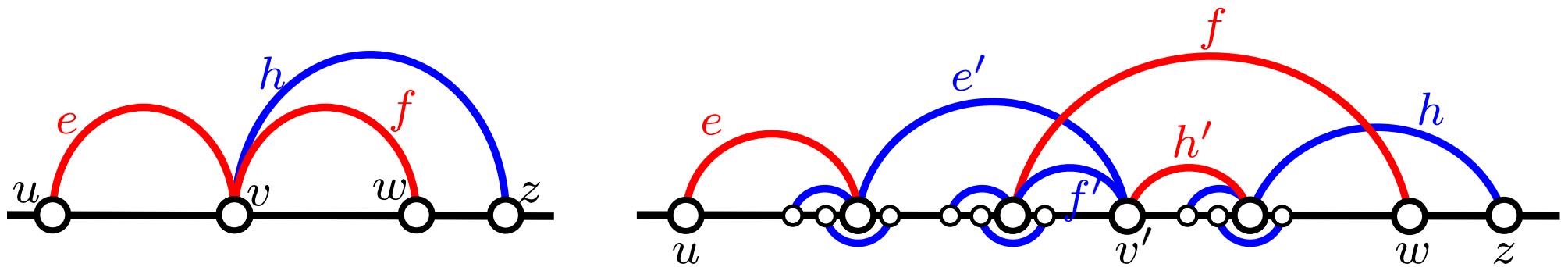


From  $G_n$  maxdeg-5 to  $G_1$  maxdeg-3

**Theorem.**

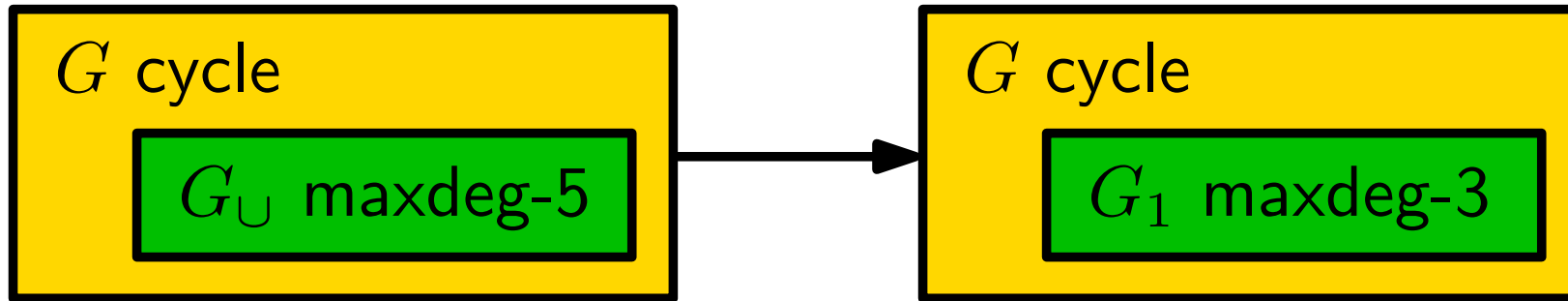


Proof:

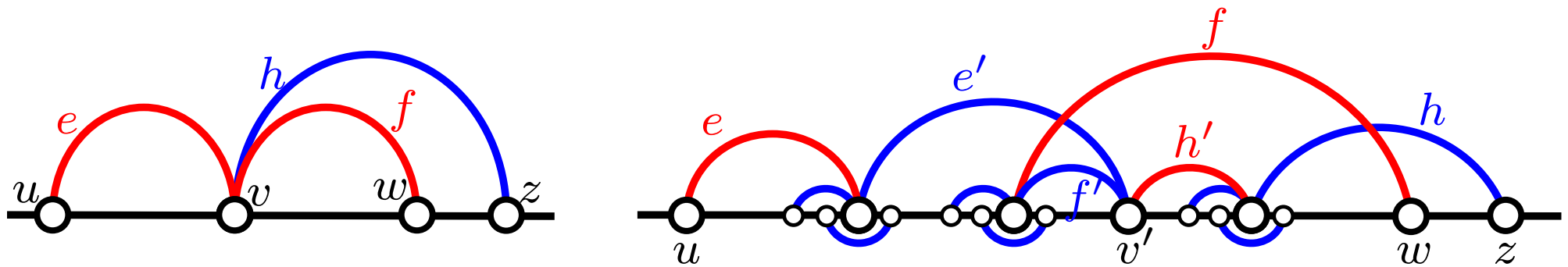


From  $G_n$  maxdeg-5 to  $G_1$  maxdeg-3

**Theorem.**



Proof:





# Our Results

reduces to



NAE-3SAT

$G$  cycle, 3 colors  
two colors maxdeg-3 + outerplanar

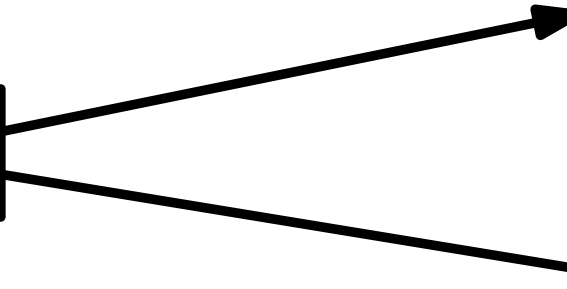
$G$  cycle + isolated vertices, 2 colors

$G$  cycle  $G_U$  maxdeg-5

$G_1$  maxdeg-3

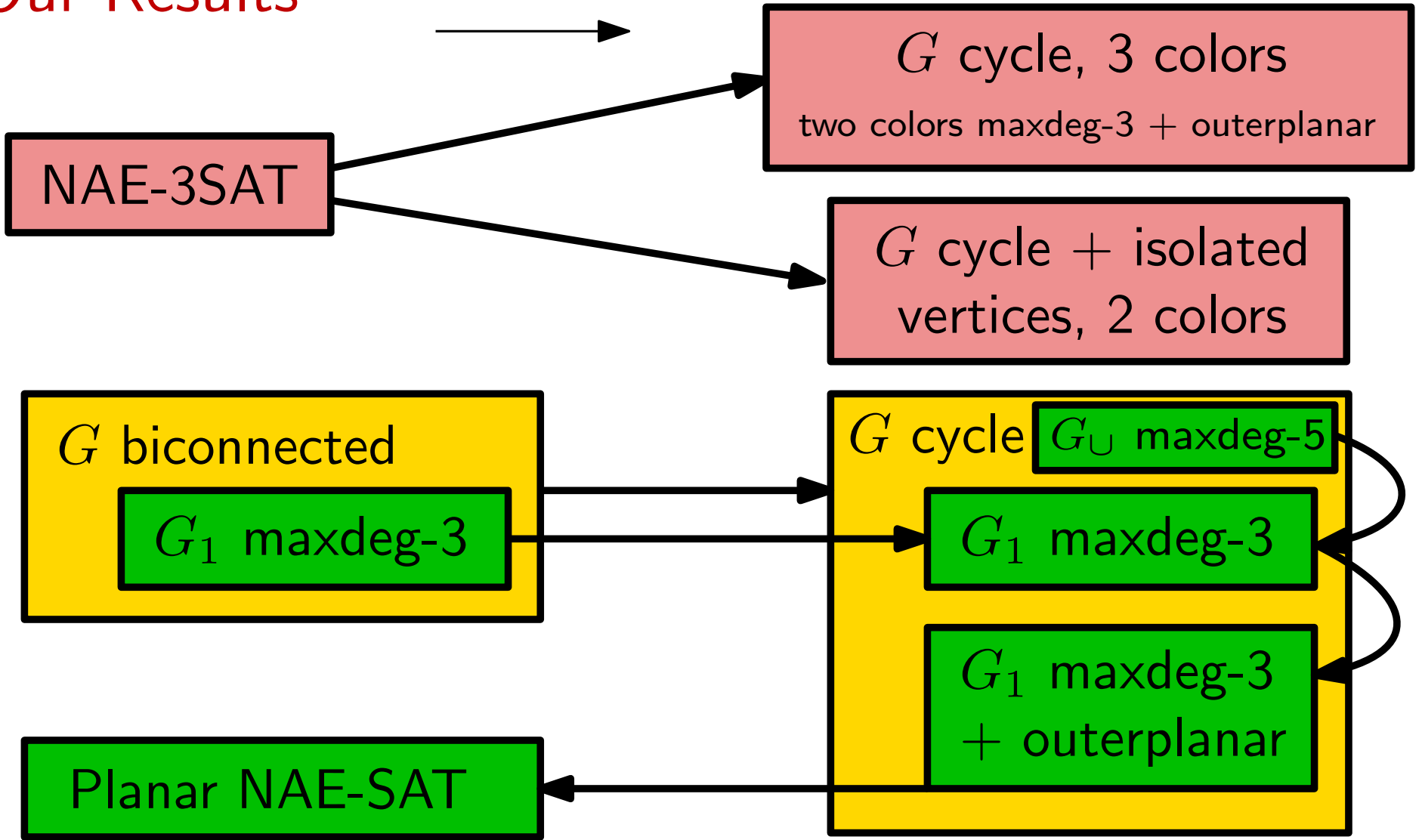
$G_1$  maxdeg-3 + outerplanar

Planar NAE-SAT



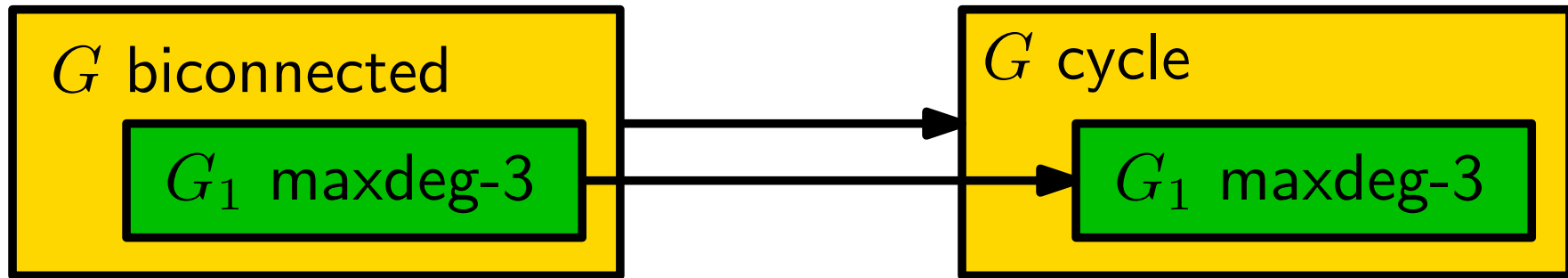
# Our Results

reduces to



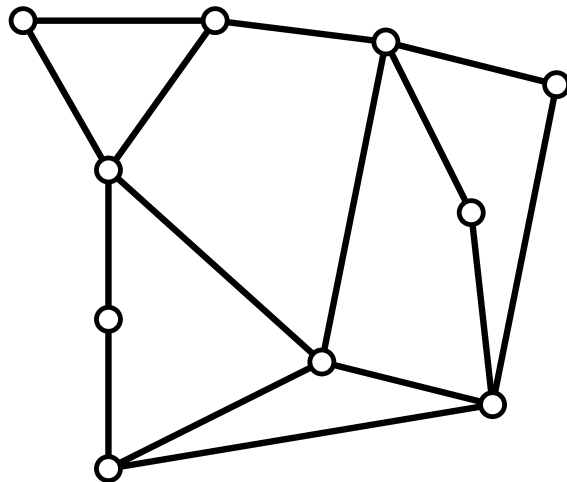
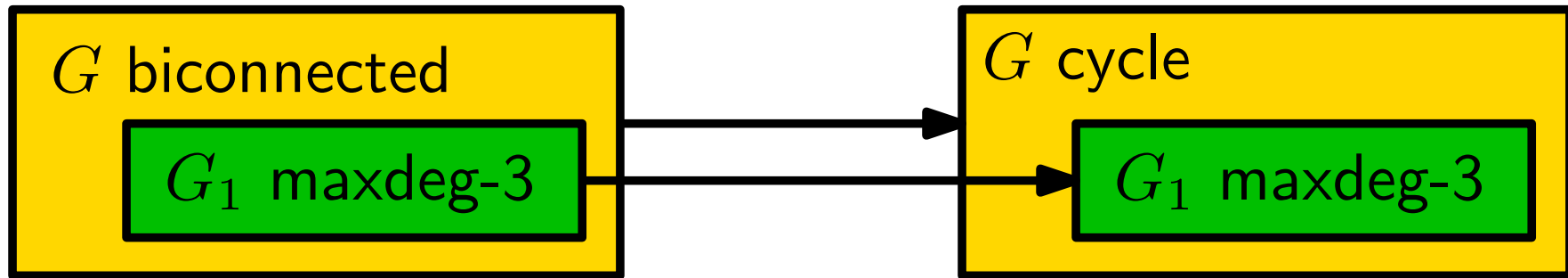
# From biconnected to cycle

**Theorem.**



# From biconnected to cycle

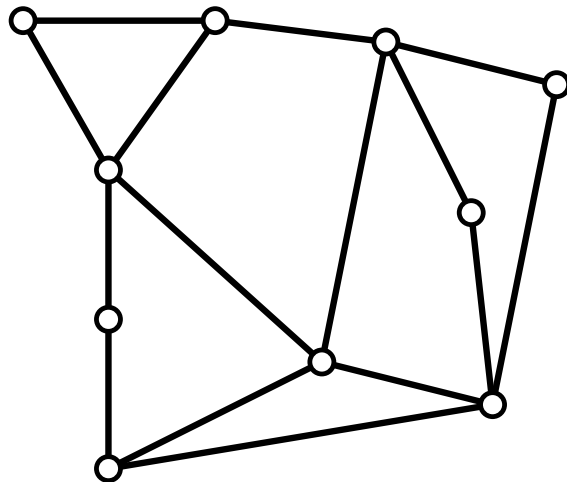
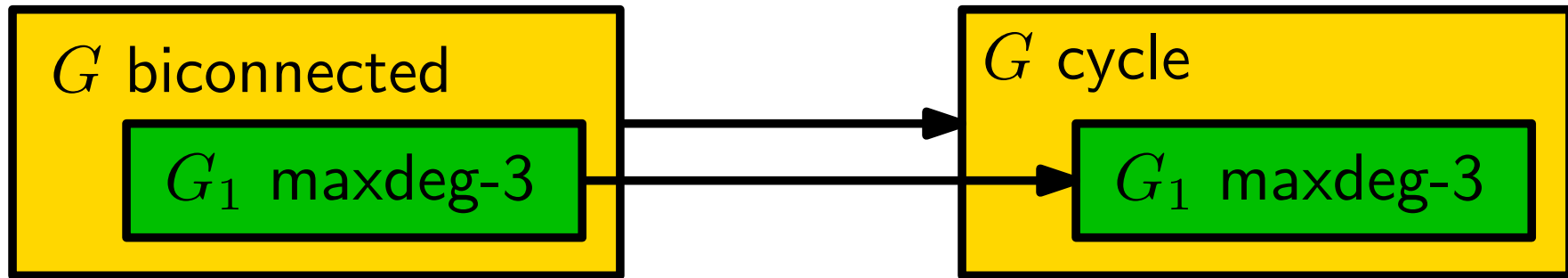
**Theorem.**



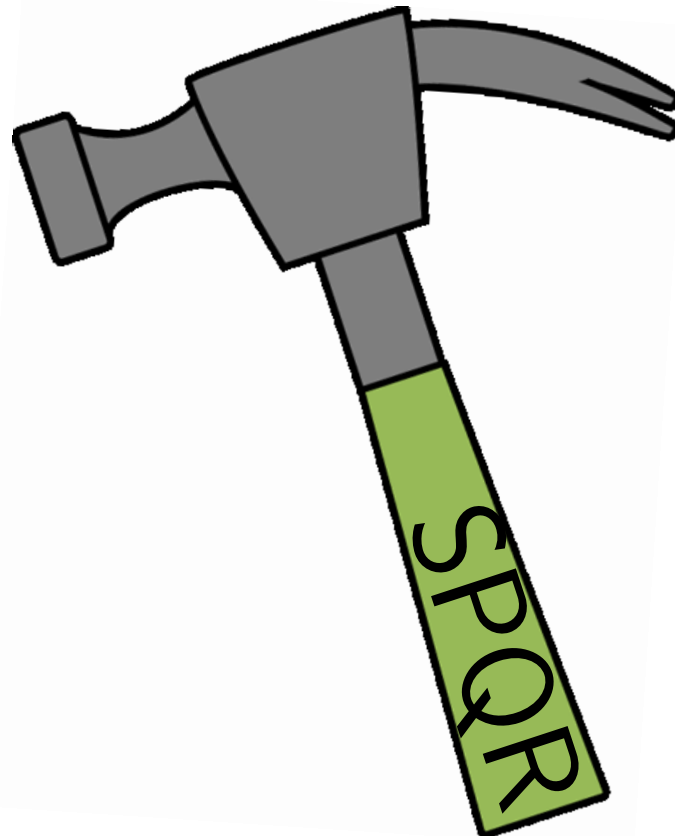
$G$  biconnected

# From biconnected to cycle

Theorem.

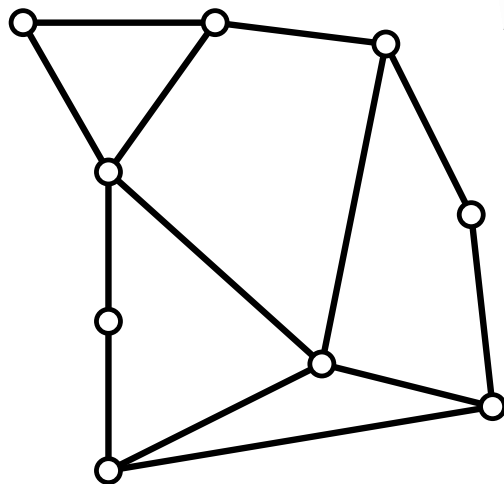
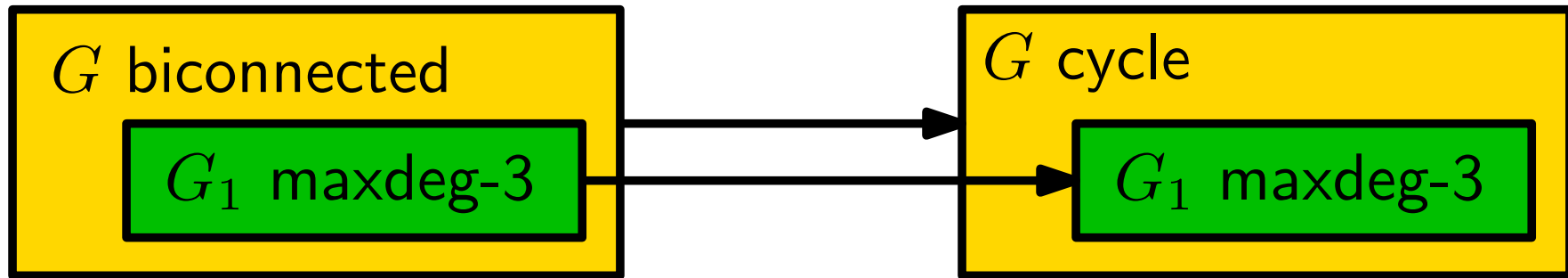


$G$  biconnected

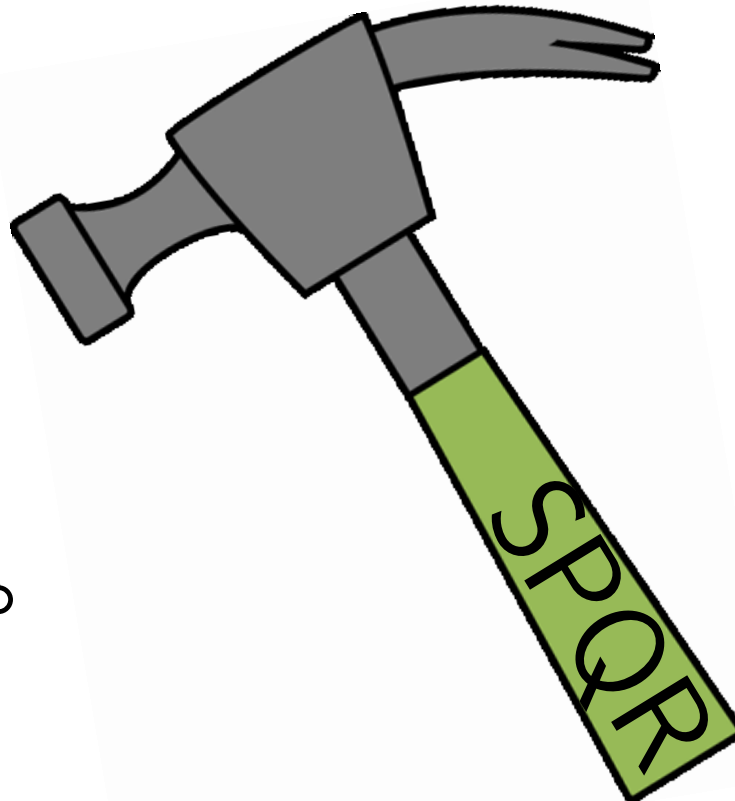


# From biconnected to cycle

Theorem.

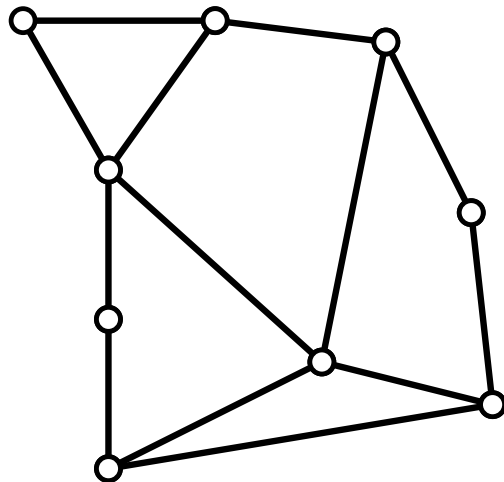
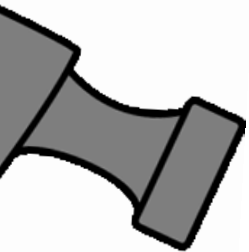
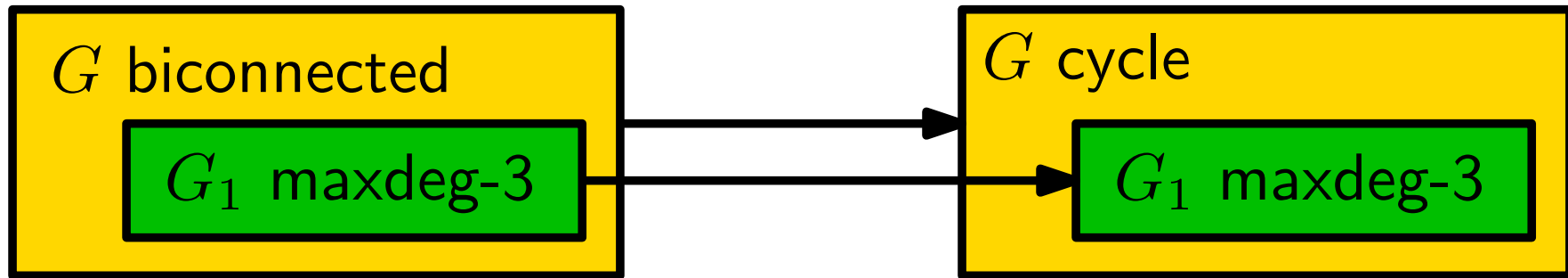


$G$  biconnected



# From biconnected to cycle

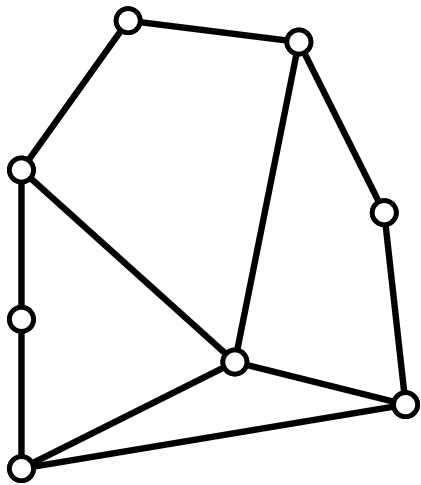
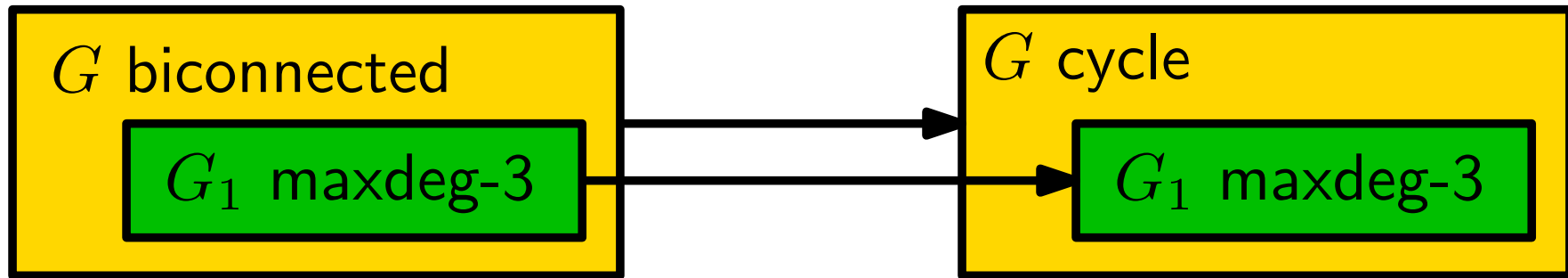
Theorem.



$G$  biconnected

# From biconnected to cycle

Theorem.

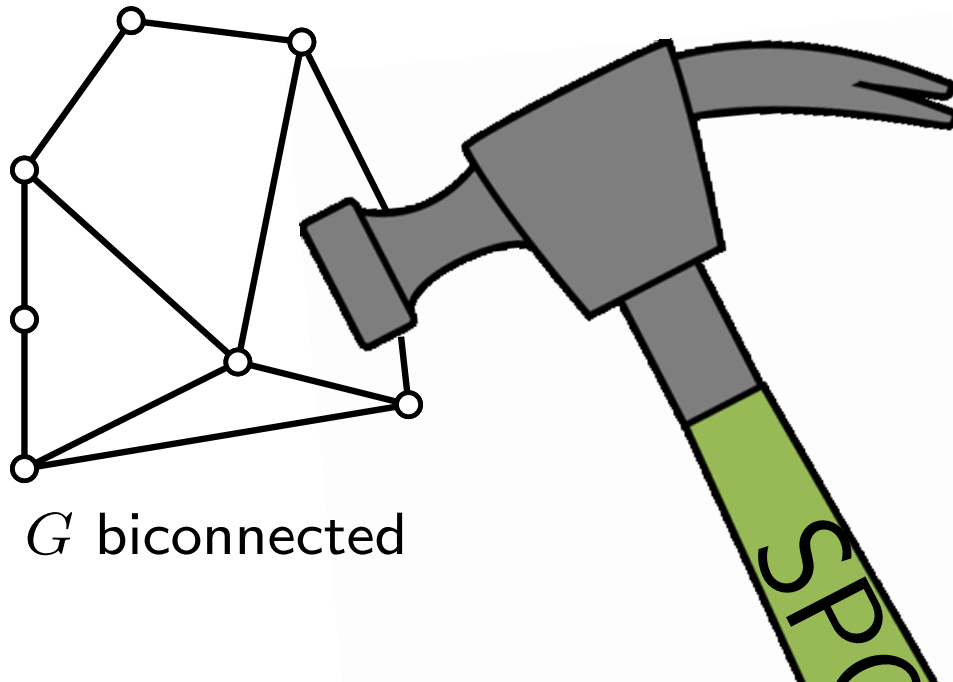
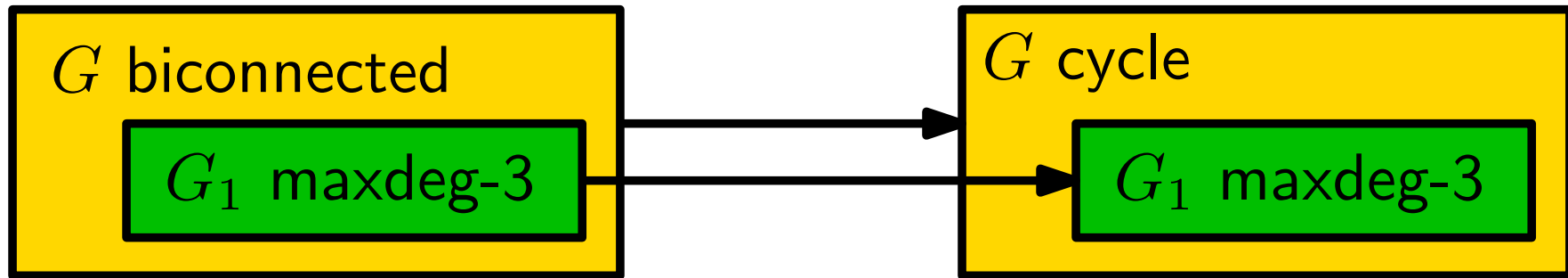


$G$  biconnected



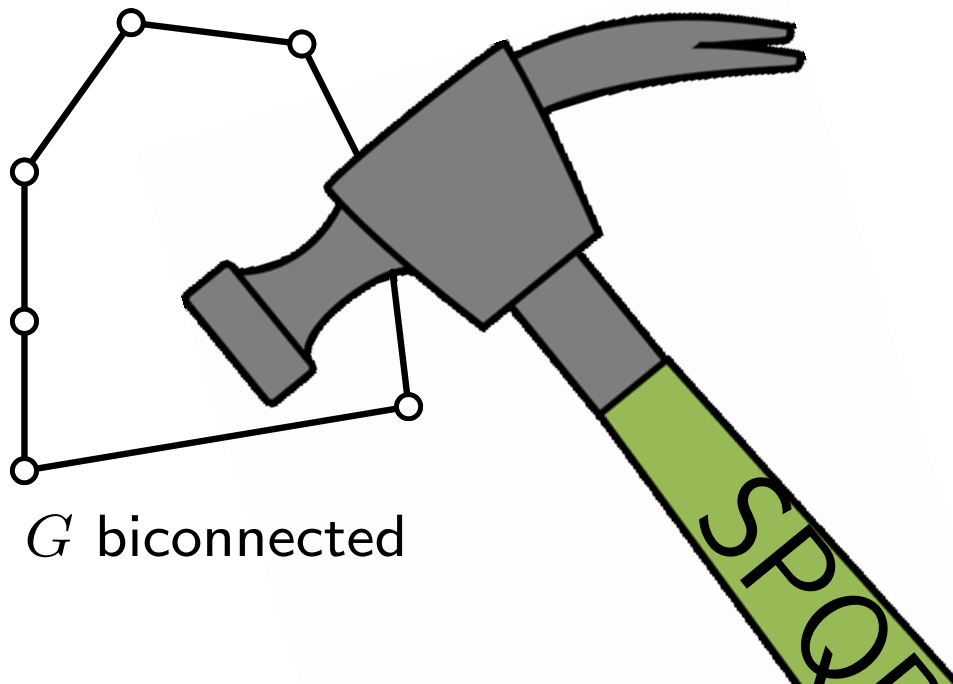
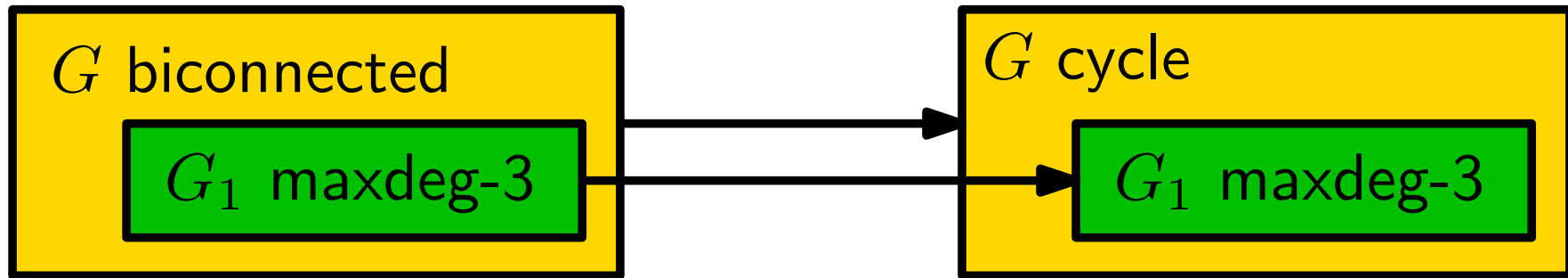
# From biconnected to cycle

Theorem.



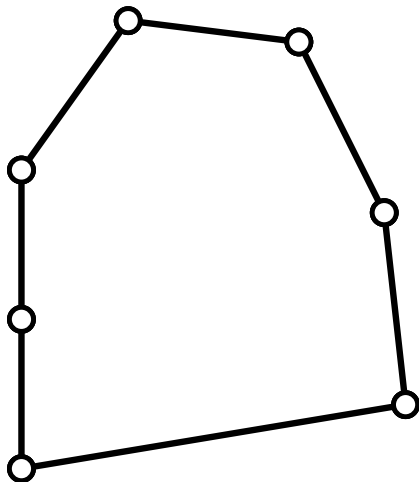
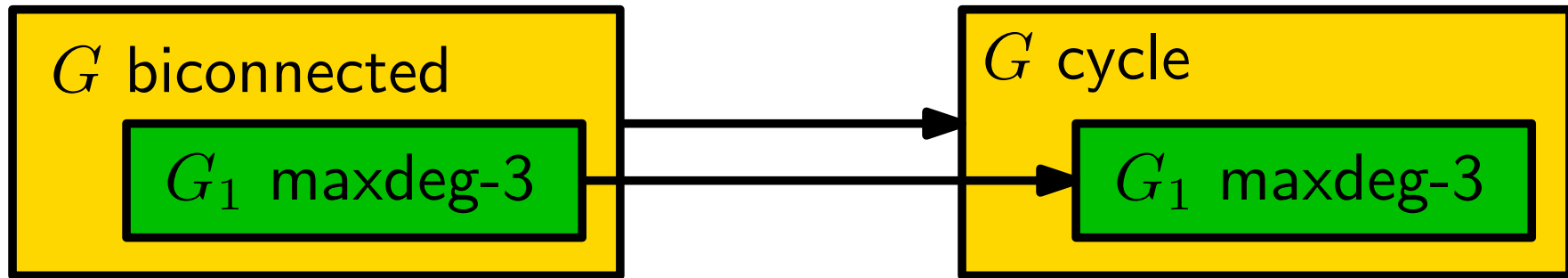
# From biconnected to cycle

Theorem.



# From biconnected to cycle

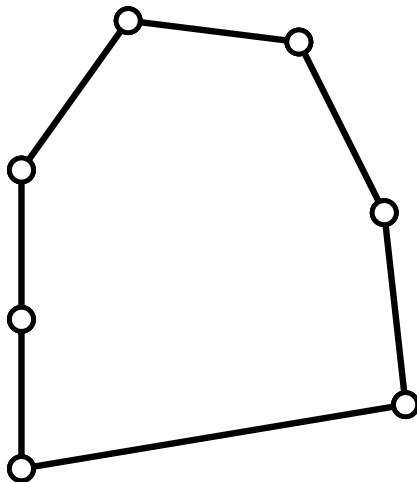
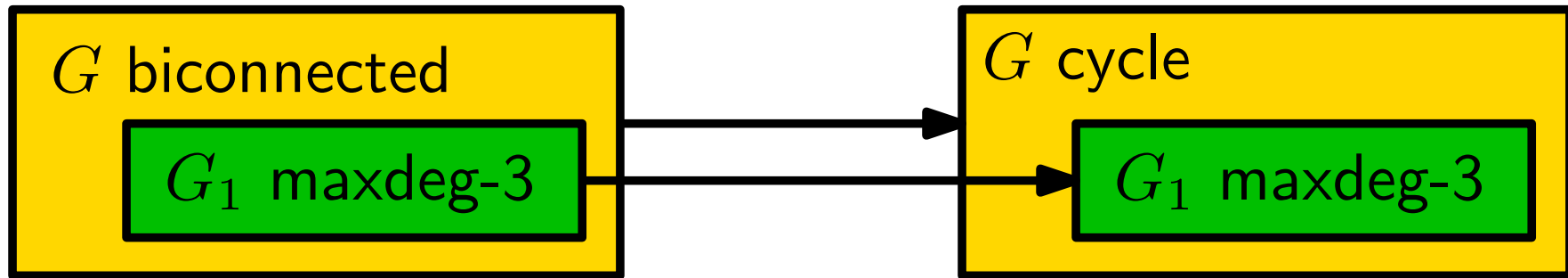
**Theorem.**



$G$  cycle

# From biconnected to cycle

**Theorem.**

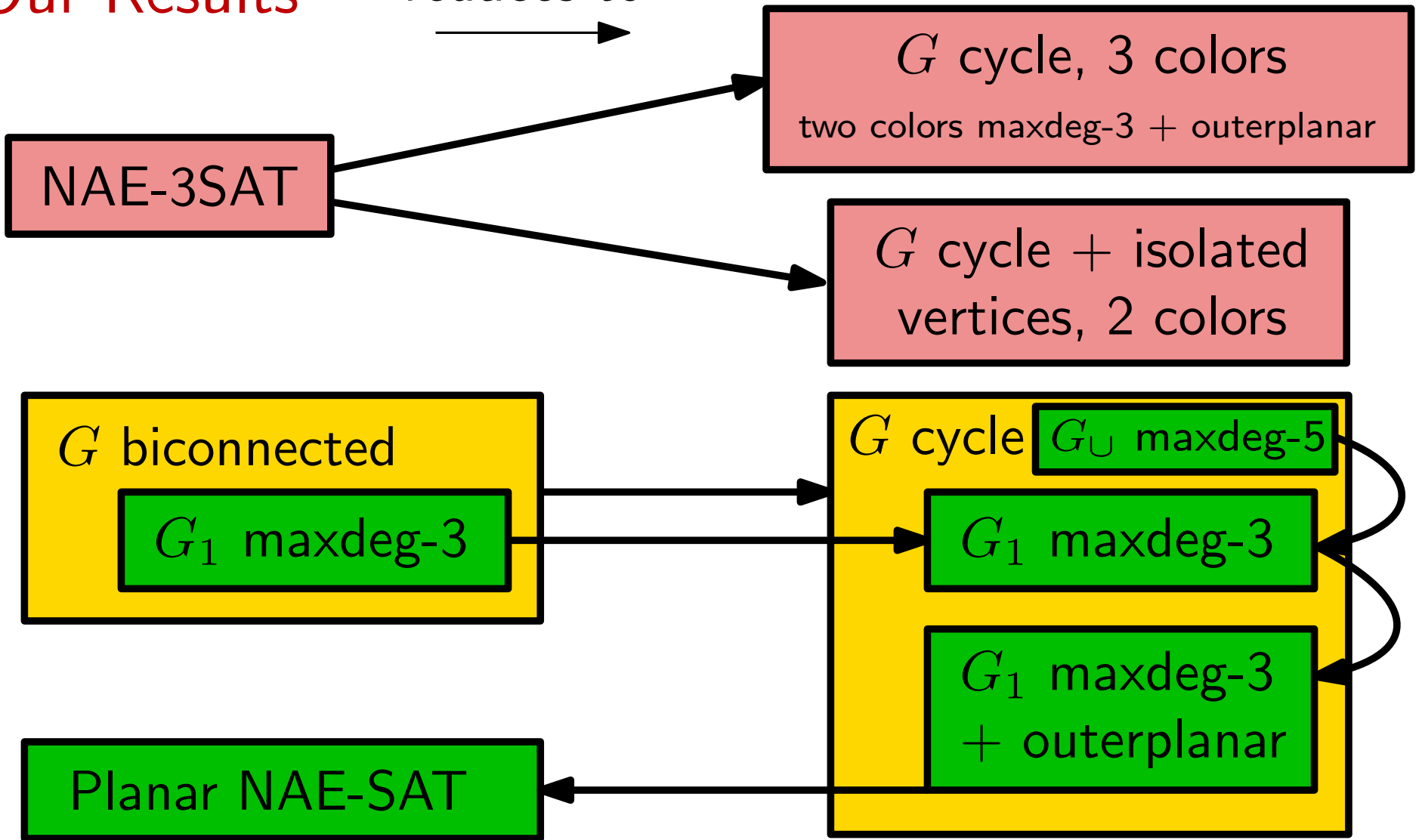


$G$  cycle



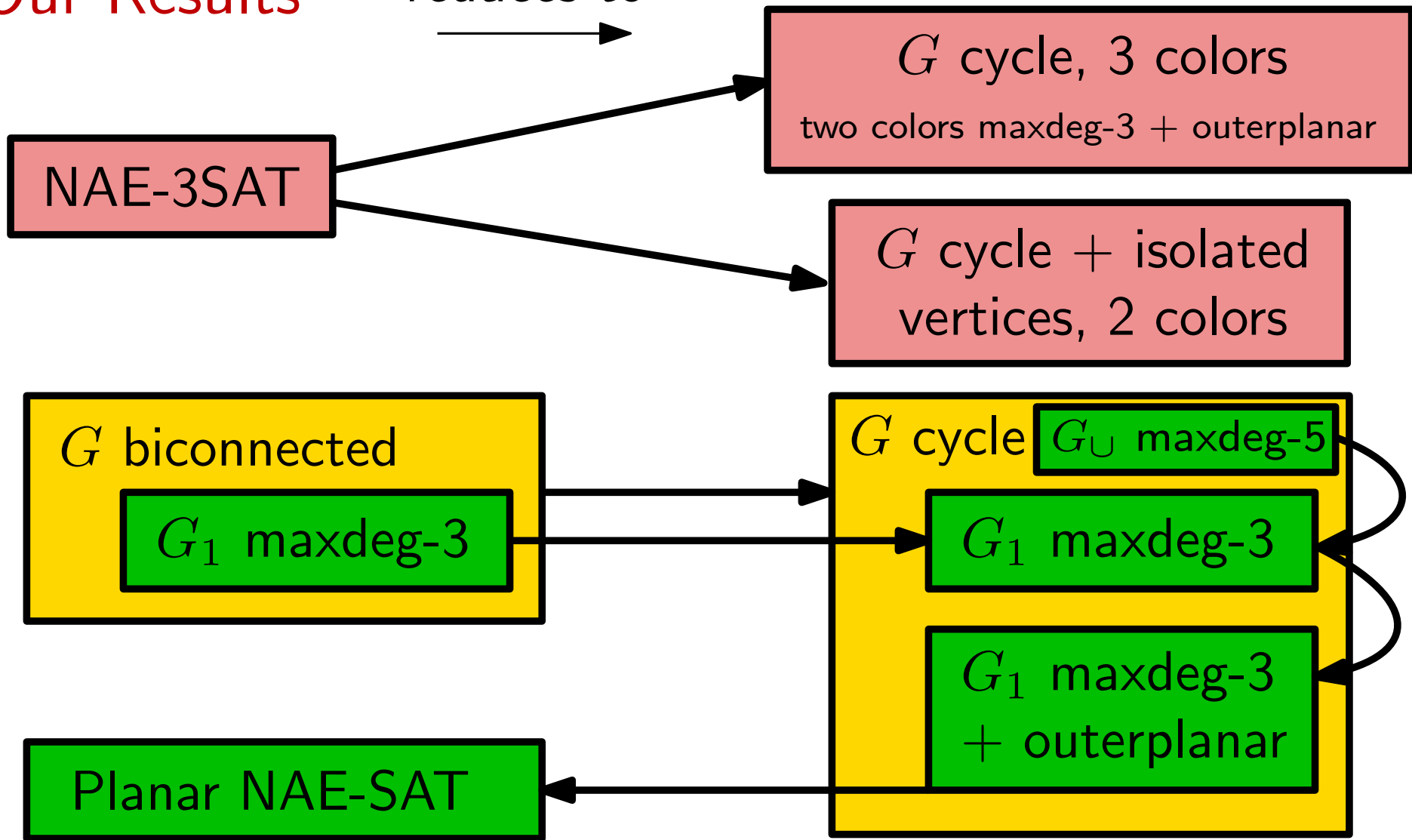
# Our Results

reduces to



# Our Results

reduces to



$G$  biconnected  $\Rightarrow$  can draw simultaneous orthogonal embedding with  $\leq 3$  bends per edge

# Drawing Algorithm

- ▶ Based on Biedl & Kant [ESA '94, Comput. Geom. '98]

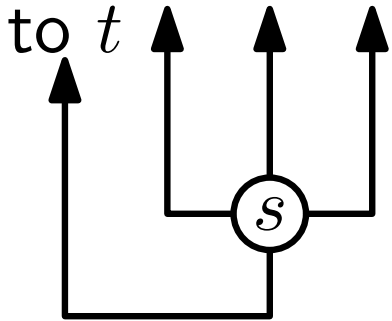
# Drawing Algorithm

- ▶ Based on Biedl & Kant [ESA '94, Comput. Geom. '98]
- ▶ Place vertices bottom-to-top by  $s$ - $t$ -ordering on  $G$



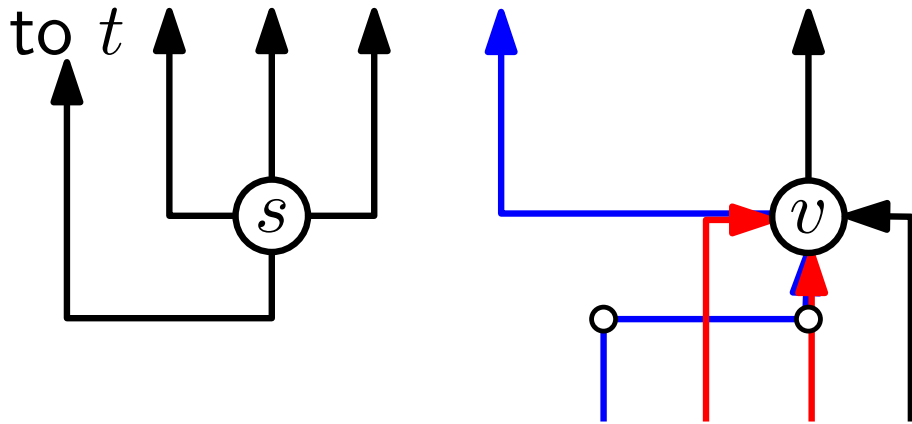
# Drawing Algorithm

- ▶ Based on Biedl & Kant [ESA '94, Comput. Geom. '98]
- ▶ Place vertices bottom-to-top by  $s$ - $t$ -ordering on  $G$



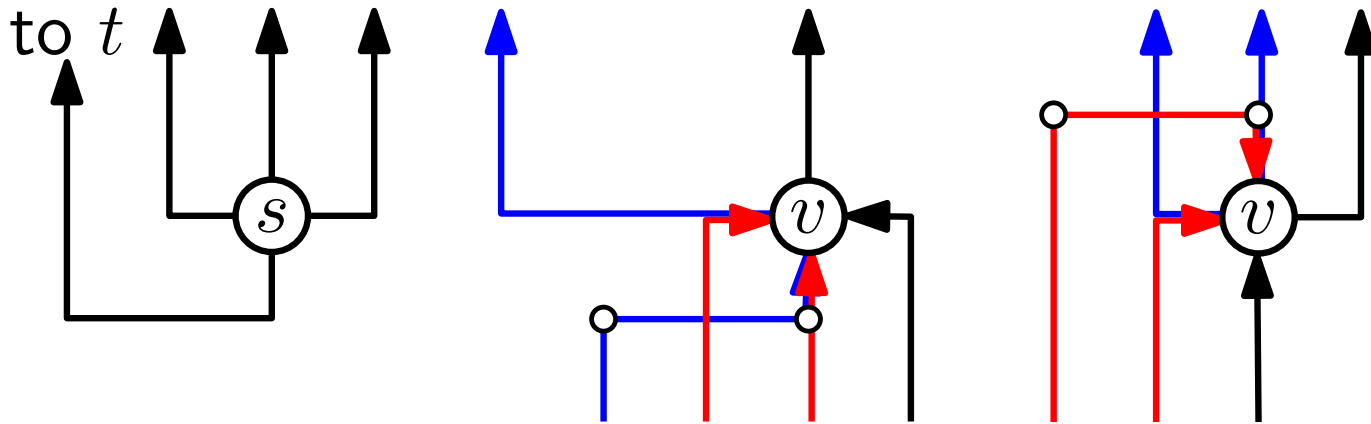
# Drawing Algorithm

- ▶ Based on Biedl & Kant [ESA '94, Comput. Geom. '98]
- ▶ Place vertices bottom-to-top by  $s$ - $t$ -ordering on  $G$



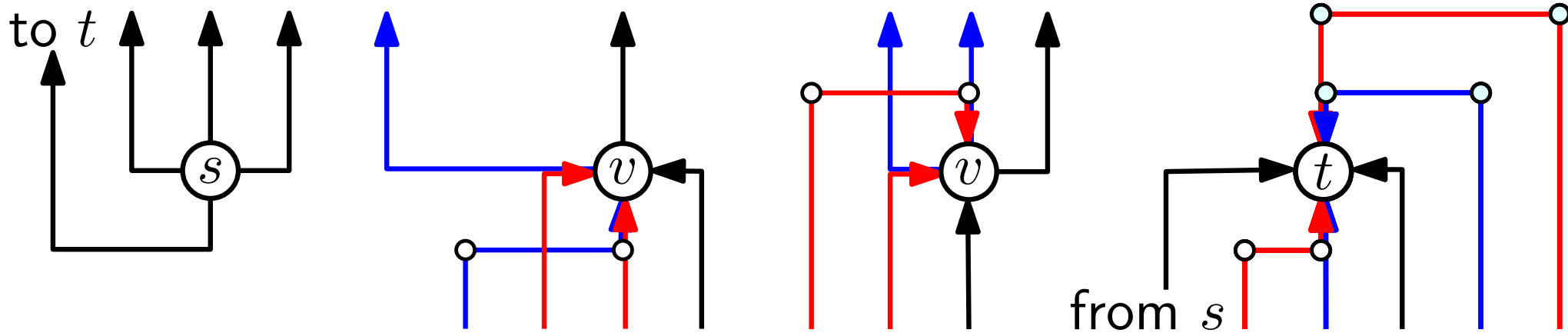
# Drawing Algorithm

- ▶ Based on Biedl & Kant [ESA '94, Comput. Geom. '98]
- ▶ Place vertices bottom-to-top by  $s$ - $t$ -ordering on  $G$



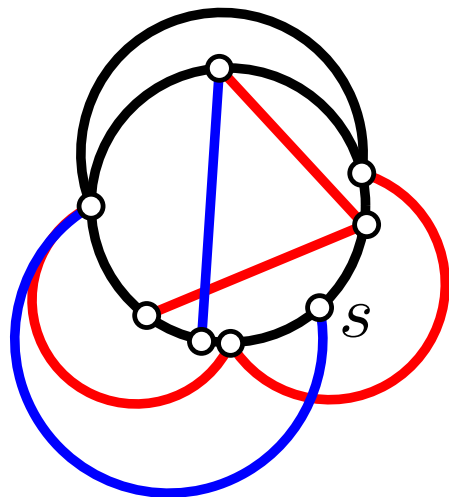
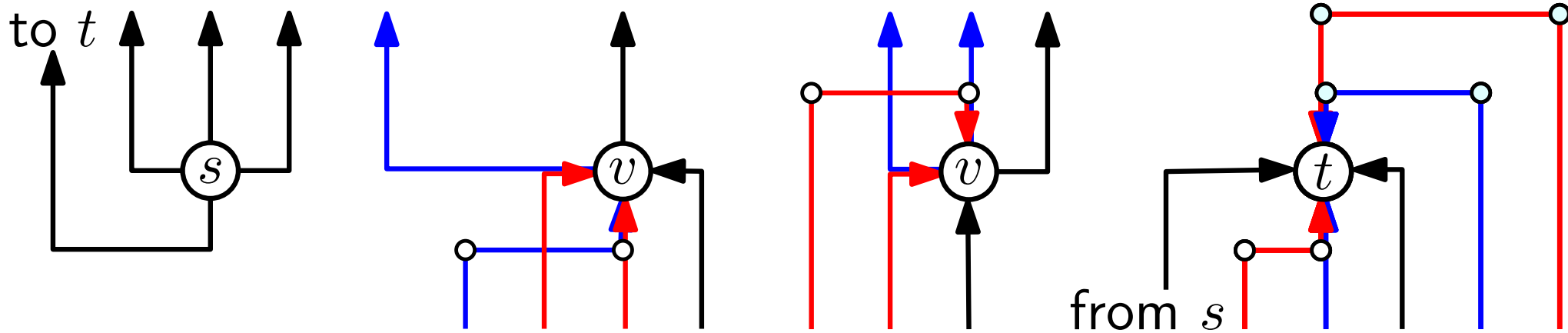
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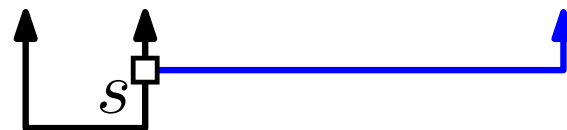
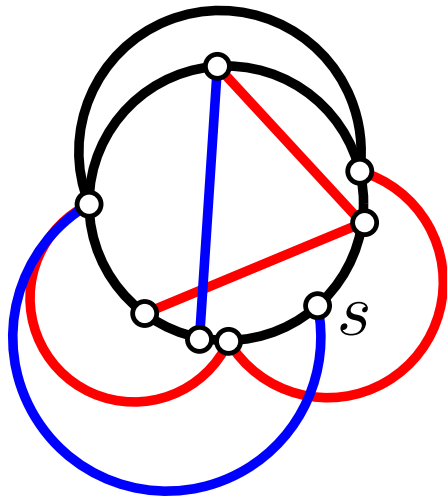
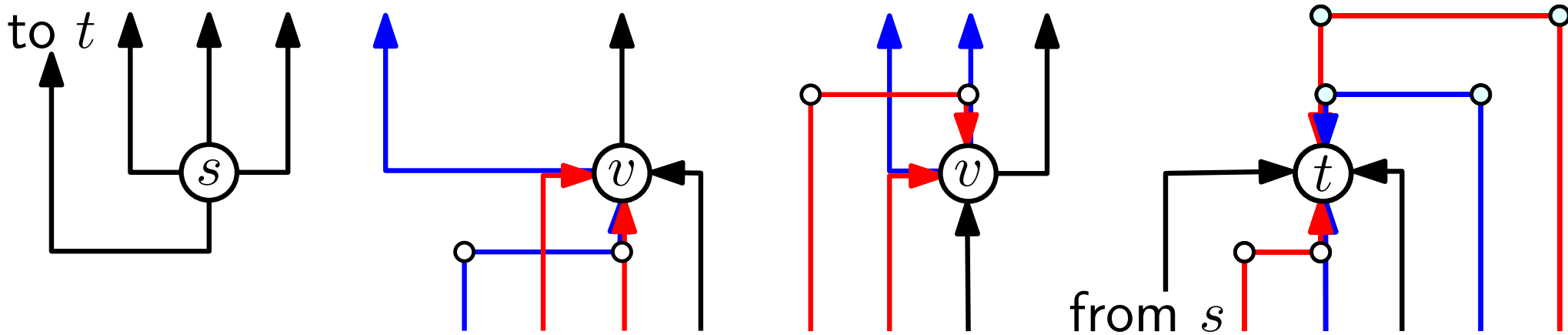
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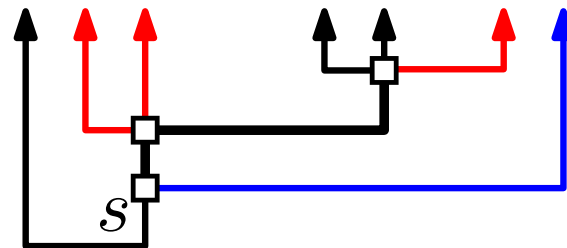
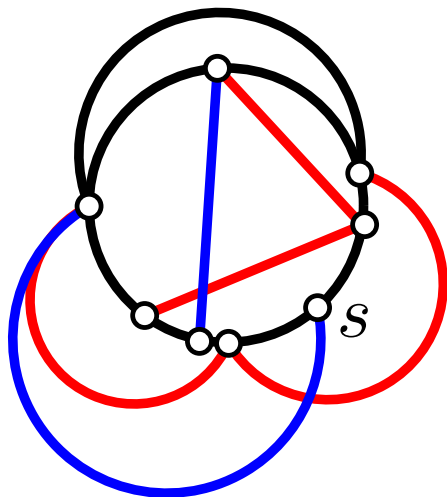
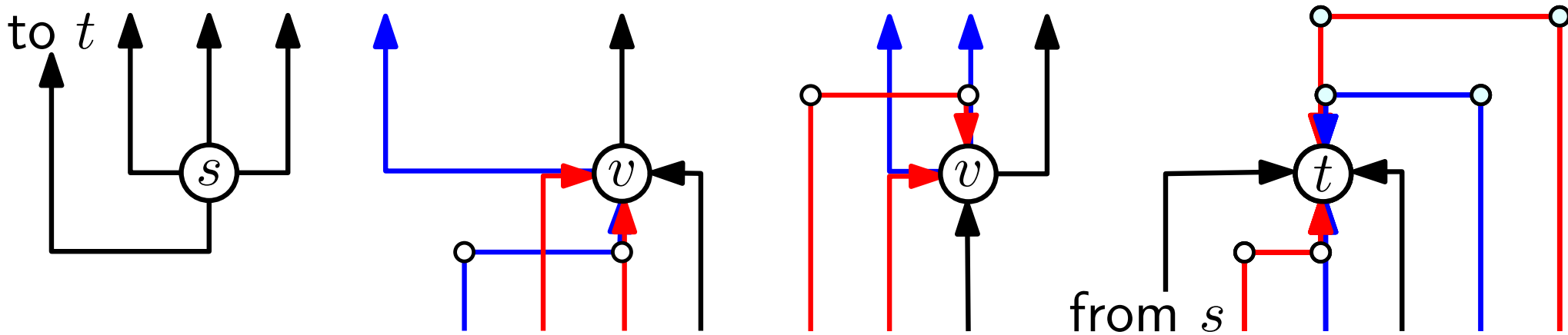
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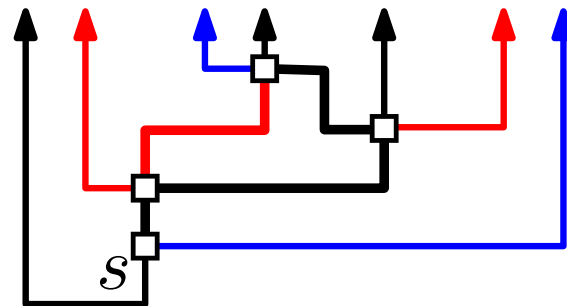
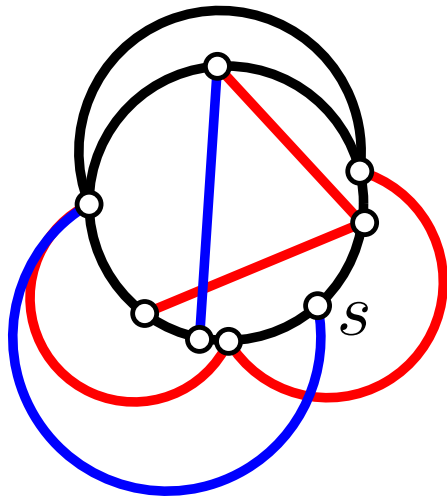
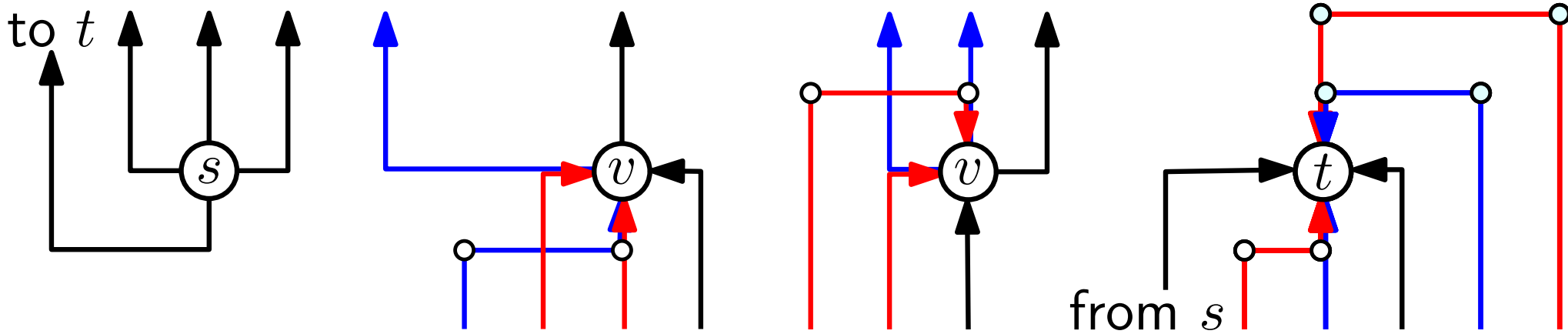
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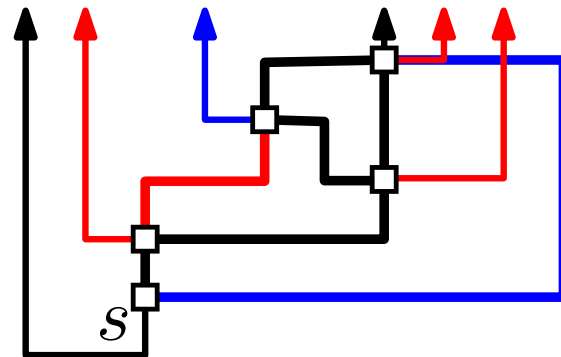
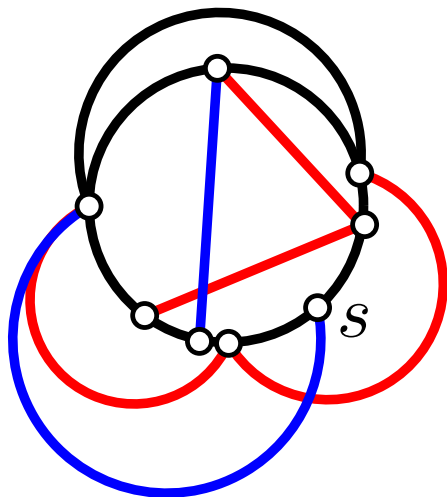
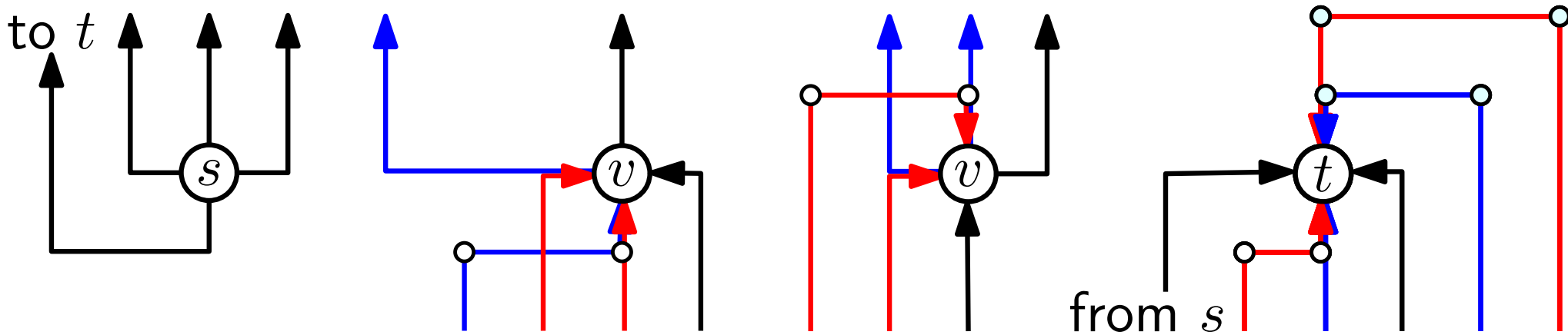
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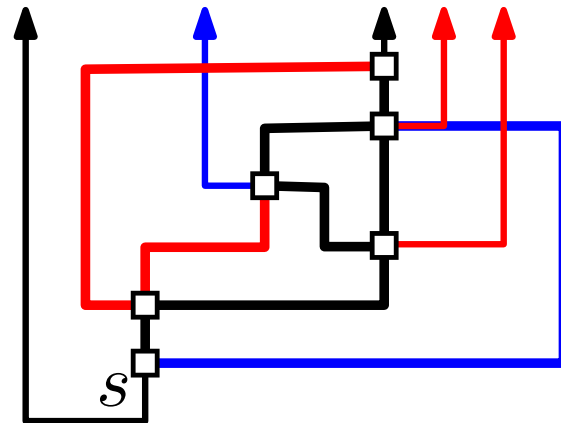
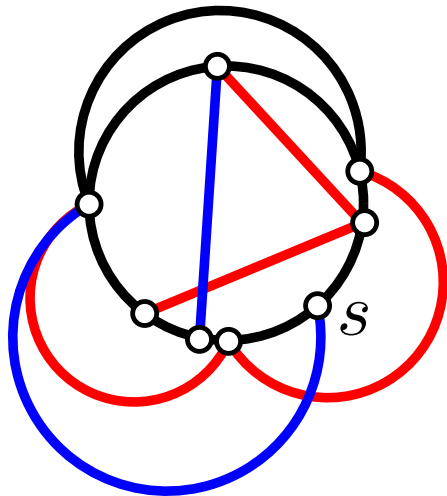
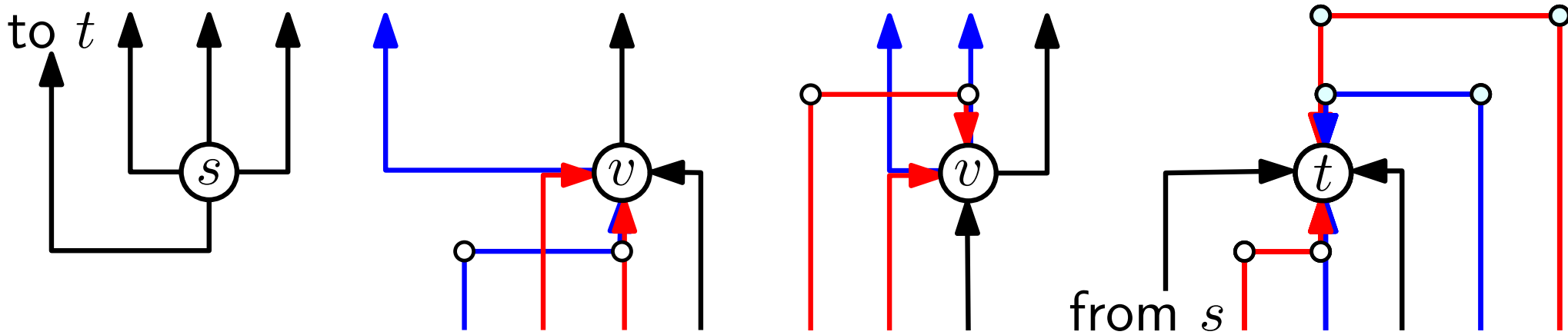
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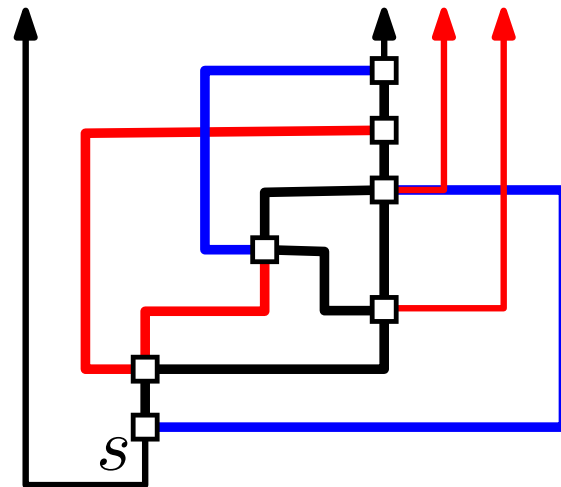
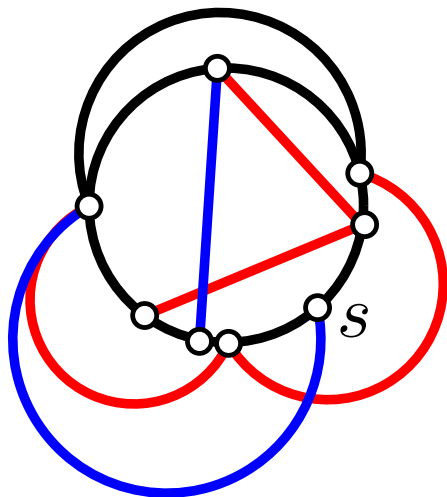
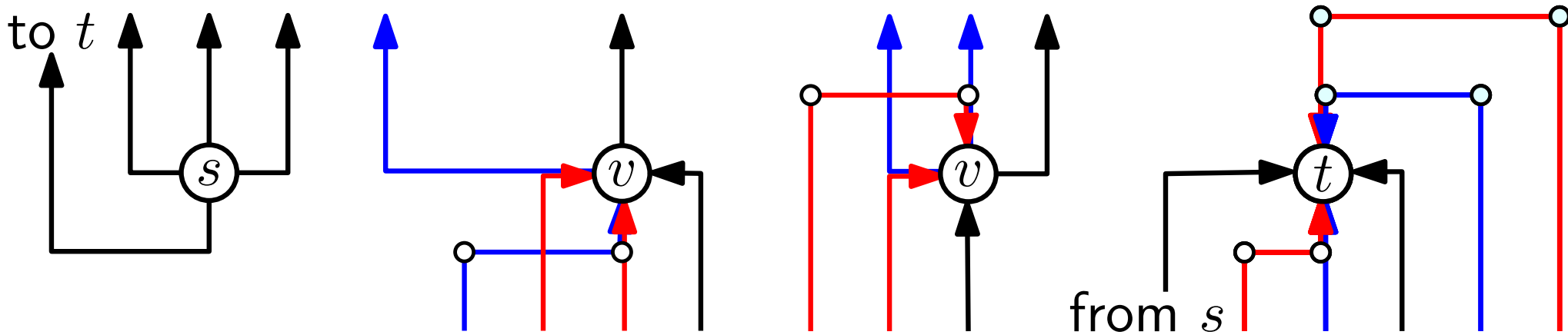
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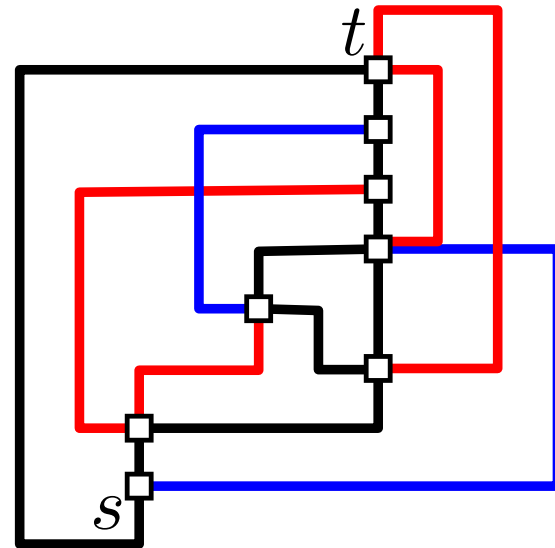
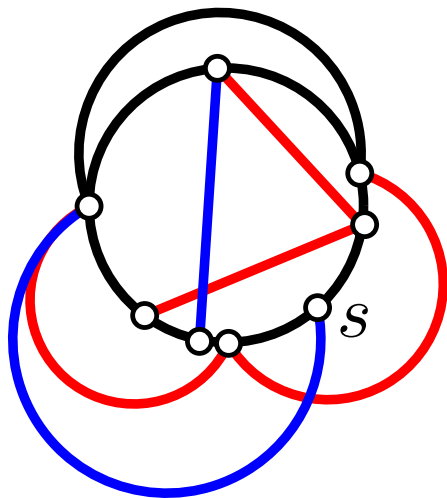
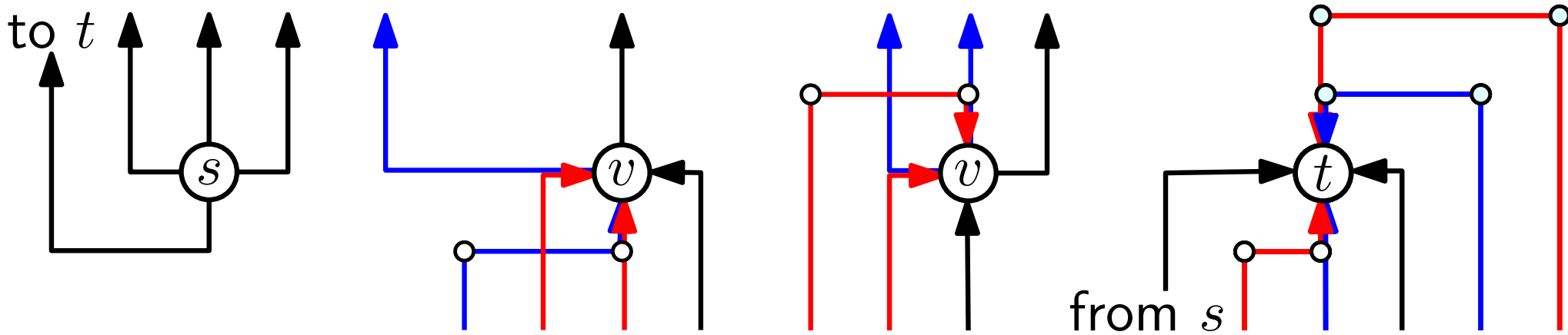
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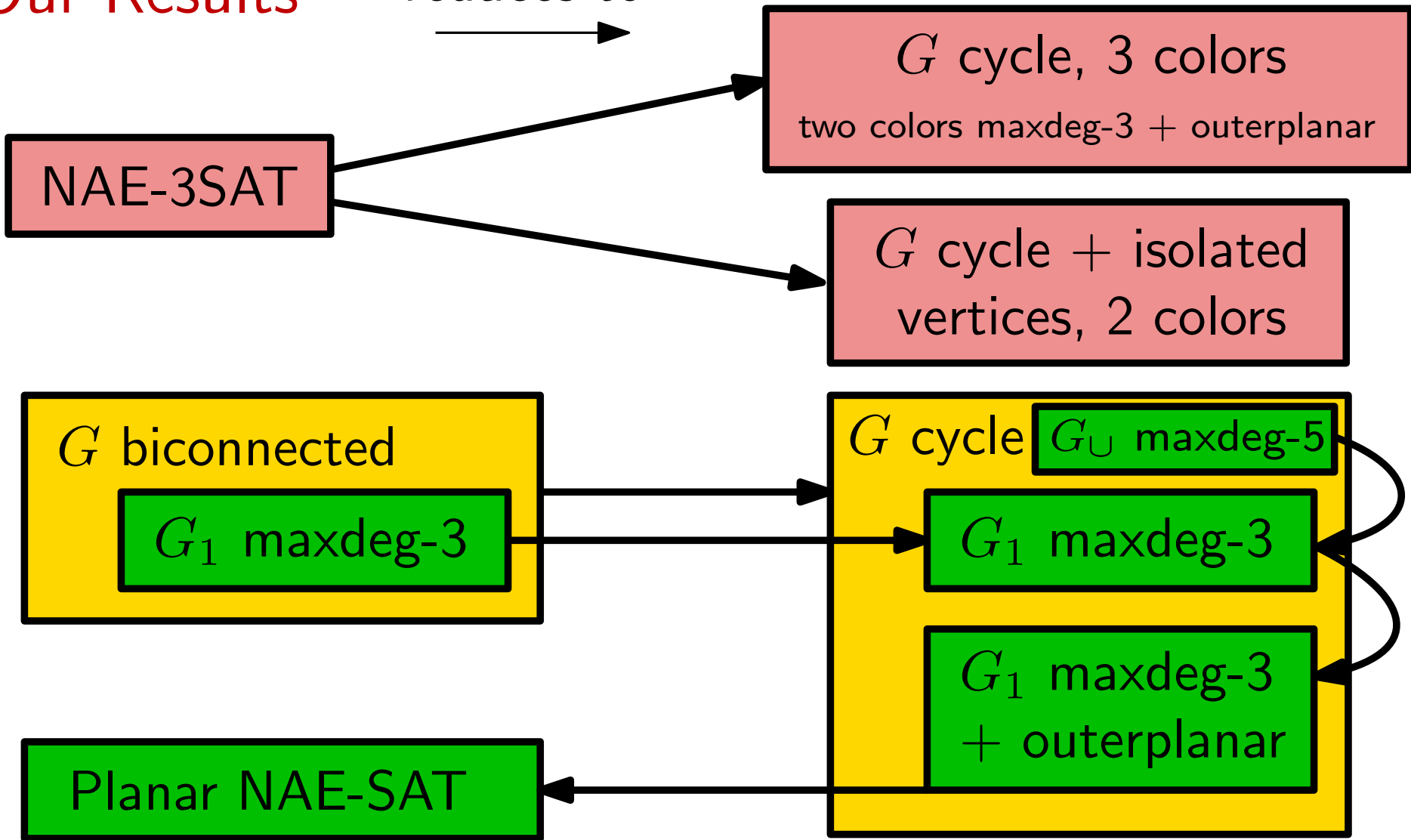
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# Our Results

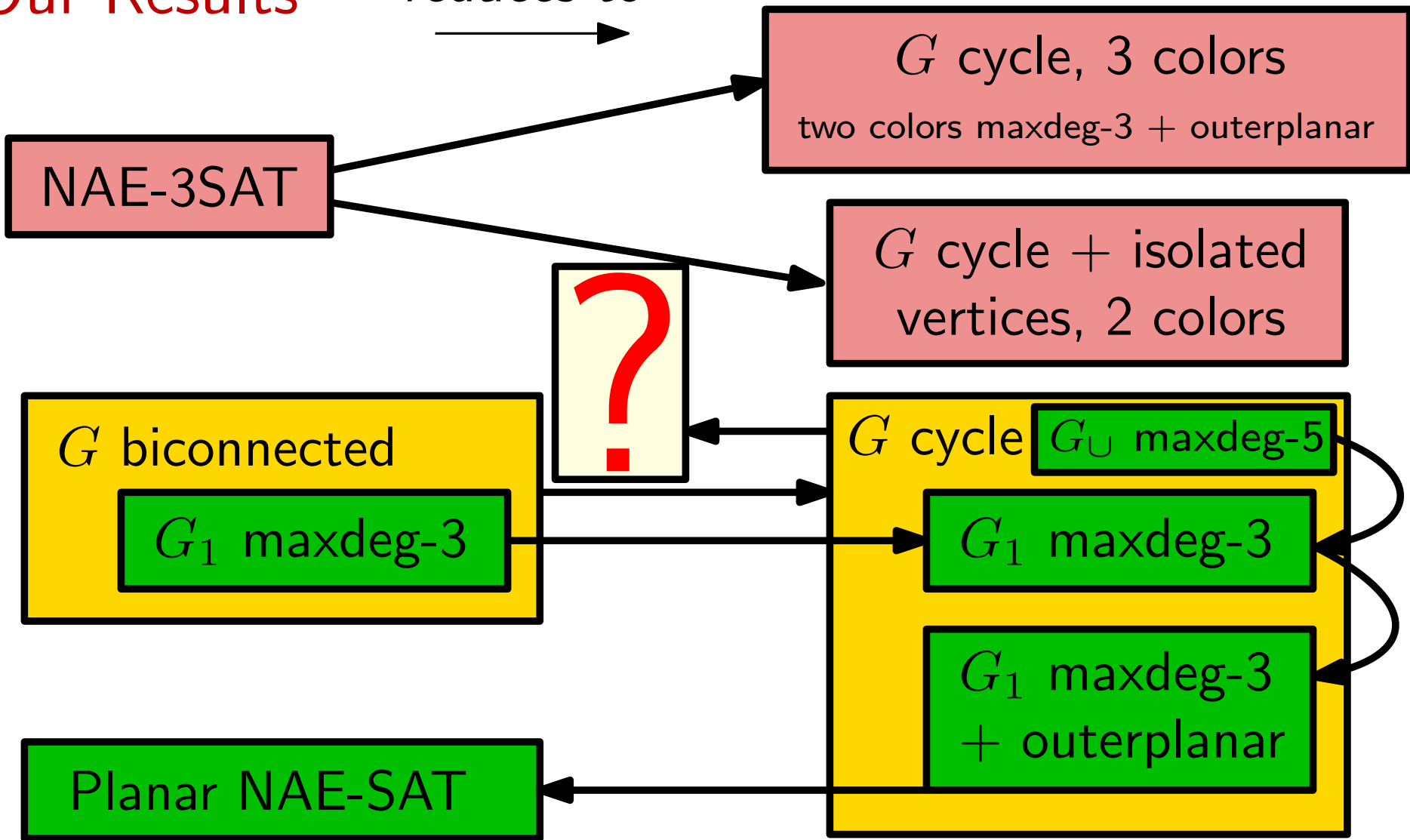
reduces to



$G$  biconnected  $\Rightarrow$  can draw simultaneous orthogonal embedding with  $\leq 3$  bends per edge

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