24TH INTERNATIONAL SYMPOSIUM ON GRAPH DRAWING AND NETWORK VISUALIZATION

STACK AND QUEUE LAYOUTS VIA LAYERED SEPARATORS

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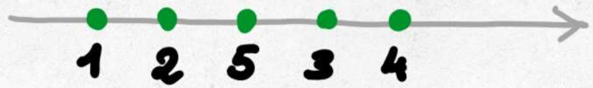
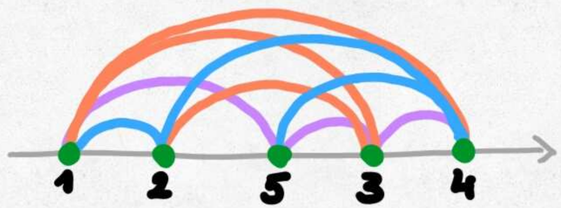
STACK LAYOUT OR BOOK EMBEDDING

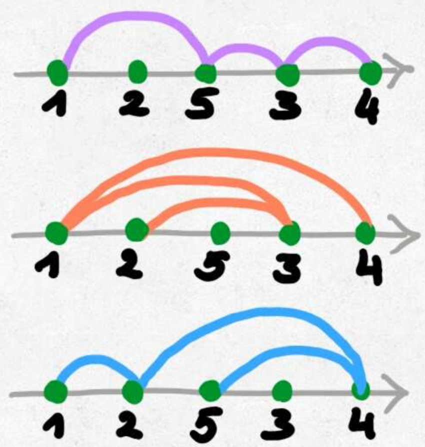
OF A GRAPH $G=(V,E)$ CONSISTS OF:

1) A TOTAL ORDER OF V



2) A PARTITION OF E INTO K SETS, CALLED STACKS OR PAGES, SUCH THAT NO TWO EDGES IN THE SAME STACK CROSS.





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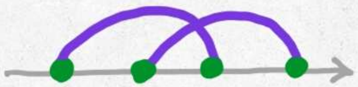


STACK NUMBER OR BOOK THICKNESS OR PAGE NUMBER OR FIXED OUTERTHICKNESS

OF A GRAPH $G=(V,E)$ IS THE MINIMUM K SUCH THAT G ADMITS A STACK LAYOUT WITH K STACKS.

STACK NUMBER OF GRAPH CLASSES

OUTERPLANAR GRAPHS : AT MOST 1

SUB-HAMILTONIAN PLANAR GRAPHS : AT MOST 2

3-TREES : AT MOST 3 [HEATH 1984]

PLANAR GRAPHS : AT MOST 4 [YANNAKAKIS 1989]

GENUS-g GRAPHS : AT MOST $O(\sqrt{g})$ [MALITZ 1994]

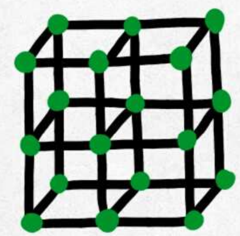
TREEWIDTH-t GRAPHS : AT MOST $O(t)$ [GANLEY, HEATH 2001]

MINOR-CLOSED FAMILIES OF GRAPHS : BOUNDED [BLANKENSHIP, OPOROWSKI 2003]

CAN WE GO BEYOND MINOR-CLOSED GRAPH FAMILIES ?

(g,k)- PLANAR GRAPHS

A GRAPH IS (g,k)-PLANAR IF IT CAN BE DRAWN ON A SURFACE WITH EULER GENUS g WITH AT MOST k CROSSINGS PER EDGE.



- FOR EVERY $g \geq 0$ AND $k \geq 1$ THERE EXIST FAMILIES OF (g,k)-PLANAR GRAPHS WHICH ARE NOT CLOSED UNDER TAKING MINORS.
- THE STACK NUMBER OF (g,k)-PLANAR GRAPHS IS $f(g,k) \cdot O(\sqrt{n})$ [MALITZ 1994; PACH, TÓTH 1997]
- THE STACK NUMBER OF $(0,1)$-PLANAR GRAPHS IS $O(1)$ [BEKOS, BRUCKDORFER, KAUFMANN, RAFTOPOULOU 2015; ALAM, BRANDENBURG, KOBOUROV 2015]

THEOREM [THIS PAPER] EVERY (g,k)-PLANAR GRAPH HAS STACK NUMBER

$$f(g,k) \cdot O(\log n)$$

(g, d)-MAP GRAPHS

A GRAPH IS A (g,d)-MAP GRAPH IF IT CAN BE REPRESENTED ON A SURFACE WITH EULER GENUS g SO THAT VERTICES CORRESPOND TO "NATIONS" AND EDGES CORRESPOND TO ADJACENCIES BETWEEN NATIONS, ALSO VIA SINGLE POINTS, SO THAT EACH POINT IS SHARED BY AT MOST d NATIONS.

- FOR EVERY $g \geq 0$ AND $d \geq 4$ THERE EXIST FAMILIES OF (g,d)-MAP GRAPHS WHICH ARE NOT CLOSED UNDER TAKING MINORS.
- THE STACK NUMBER OF (g,d)-MAP GRAPHS IS $f(g,d) \cdot O(\sqrt{n})$ [MALITZ 1994; CHEN, GRIGNI, PAPADIMITRIOU 2002]

THEOREM [THIS PAPER] EVERY (g,d)-MAP GRAPH HAS STACK NUMBER

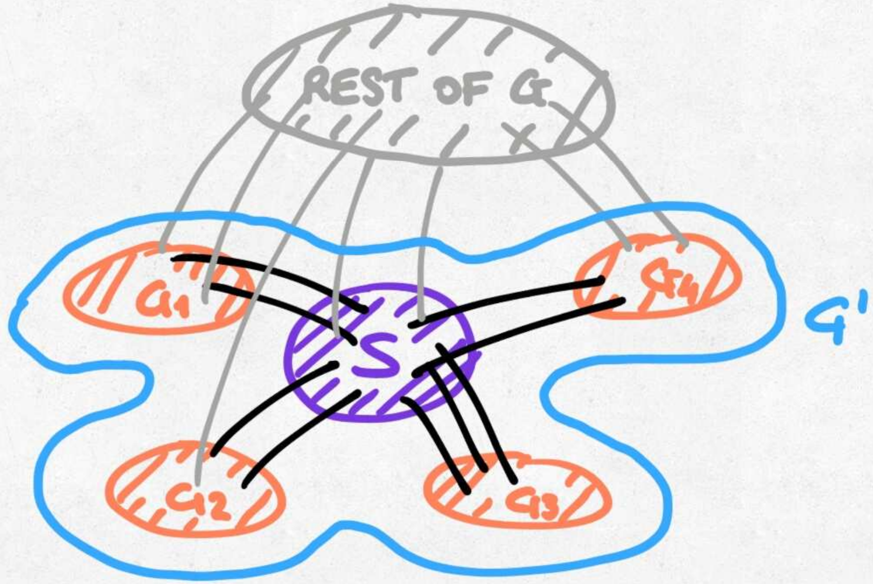
$$f(g,d) \cdot O(\log n)$$

OUR TOOL:

LAYERED SEPARATORS

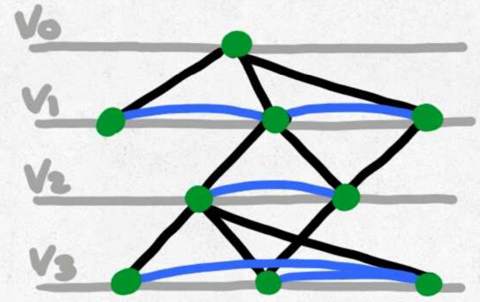
SEPARATORS

A GRAPH $G=(V,E)$ ADMITS AN ℓ-SEPARATOR IF, FOR EVERY SUBGRAPH $G'=(V',E')$ OF G, THERE EXISTS A SET $S \subseteq V'$ SUCH THAT $|S| \leq \ell$ AND SUCH THAT EACH CONNECTED COMPONENT G_i OF $G'-S$ HAS AT MOST $\frac{|V'|}{2}$ VERTICES.



LAYERING

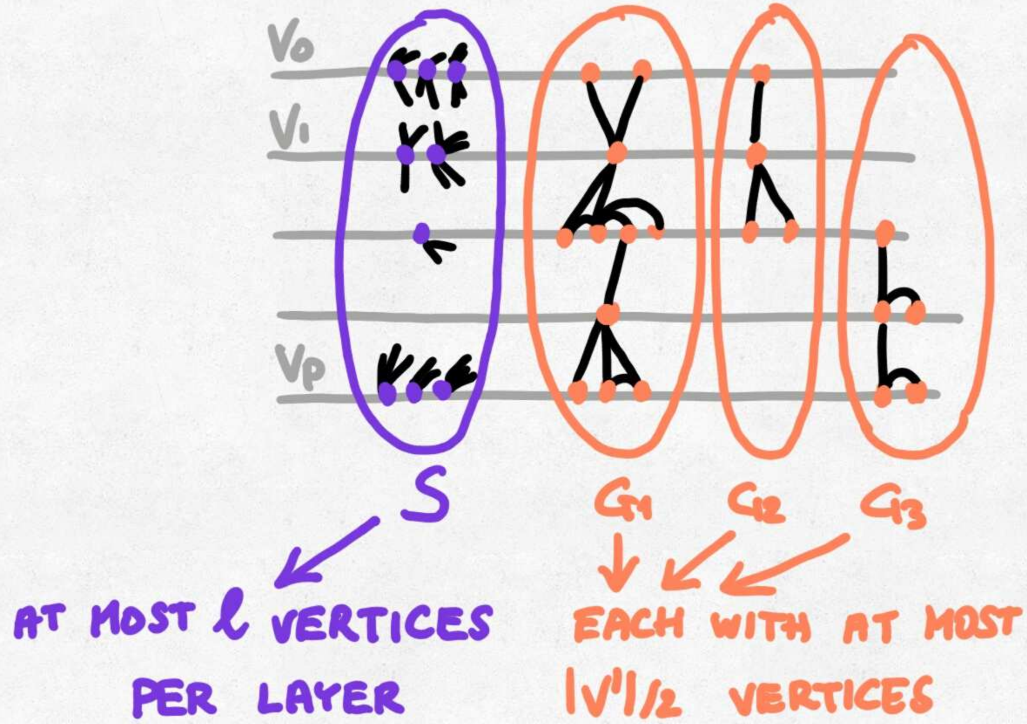
GIVEN $G = (V, E)$ A LAYERING OF G IS A PARTITION OF V INTO SETS $V_0, V_1, \ldots, V_p$, CALLED LAYERS, SUCH THAT EVERY EDGE IN E HAS ITS END-VERTICES IN THE SAME LAYER (IT IS AN INTRA-LAYER EDGE) OR IN CONSECUTIVE LAYERS (IT IS AN INTER-LAYER EDGE).



OBSERVATION : EVERY GRAPH HAS A LAYERING.

LAYERED SEPARATORS

A GRAPH $G=(V,E)$ ADMITS A LAYERED ℓ-SEPARATOR IF THERE EXISTS A LAYERING $V_0, V_1, \ldots, V_p$ SUCH THAT, FOR EVERY SUBGRAPH $G'=(V',E')$ OF G, THERE EXISTS A SET $S \subseteq V'$ SUCH THAT $|S \cap V_i| \leq \ell$ FOR EVERY $i=0,1,\ldots,p$ AND SUCH THAT EACH CONNECTED COMPONENT G_i OF $G'-S$ HAS AT MOST $\frac{|V'|}{2}$ VERTICES.



APPLICATIONS OF LAYERED SEPARATORS

NONREPETITIVE COLORINGS, TRACK LAYOUTS, QUEUE LAYOUTS

[DUJMOVIĆ, FRATI, JORET, WOOD 2013; DUJMOVIĆ 2015; DUJMOVIĆ, MORIN, WOOD 2015]

MAIN THEOREM

THEOREM [THIS PAPER] IF AN n-VERTEX GRAPH ADMITS A LAYERED ℓ-SEPARATOR, THEN ITS STACK NUMBER IS AT MOST $5\ell \cdot \log_2 n$.

CONSEQUENCES:

SINCE n-VERTEX (g,k)-PLANAR GRAPHS HAVE LAYERED $(4g+6)(k+1)$-SEPARATORS [DUJMOVIĆ, EPPSTEIN, WOOD 2015], THEY HAVE STACK NUMBER $f(g,k) \cdot O(\log n)$

SINCE n-VERTEX (g,d)-MAP GRAPHS HAVE LAYERED $(2g+3)(2d+1)$-SEPARATORS [DUJMOVIĆ, EPPSTEIN, WOOD 2015], THEY HAVE STACK NUMBER $f(g,d) \cdot O(\log n)$

PROOF OF THE MAIN THEOREM

LET $G = (V, E)$ BE A GRAPH WITH A LAYERING $V_0, V_1, \dots, V_p$ SUCH THAT G ADMITS A LAYERED ℓ-SEPARATOR WRT $V_0, V_1, \dots, V_p$.

OUR RECURSIVE ALGORITHM CONSTRUCTS A VERTEX ORDERING WHICH SATISFIES THE LAYER-BY-LAYER INVARIANT: ALL THE VERTICES IN V_{i-1} PRECEDE ALL THE VERTICES IN V_i, FOR $i = 1, \dots, p$.

VERTEX ORDERING (1)

LET S BE A LAYERED ℓ-SEPARATOR OF G WRT $V_0, V_1, \dots, V_p$.
LET $S_i = S \cap V_i$ FOR $i=0,1,\dots,p$ (NOTE THAT $|S_i| \leq \ell$ FOR $i=0,1,\dots,p$).
LET ρ_i BE ANY TOTAL ORDERING OF S_i.
LET $G_1, G_2, \dots, G_k$ BE THE CONNECTED COMPONENTS OF $G-S$ (IN ANY ORDER).

FOR $j=1,\dots,k$, RECURSIVELY CONSTRUCT A VERTEX ORDERING σ_j OF G_j SATISFYING THE LAYER-BY-LAYER INVARIANT.

LET $\sigma_{j,i}$ BE σ_j RESTRICTED TO THE VERTICES IN V_i.

σ_j: $\sigma_{j,1}, \sigma_{j,2}, \dots, \sigma_{j,p}$

VERTEX ORDERING (2)

A VERTEX ORDERING OF G IS CONSTRUCTED AS FOLLOWS:

* THE VERTICES IN V_{i-1} PRECEDE THE VERTICES IN V_i FOR $i = 1, \dots, p$.

* FOR EVERY i EVEN, THE ORDER OF V_i CONSISTS OF ρ_i, THEN $\sigma_{1,i}$, THEN $\sigma_{2,i}$, ..., THEN $\sigma_{k-1,i}$, THEN $\sigma_{k,i}$.

* FOR EVERY i ODD, THE ORDER OF V_i CONSISTS OF ρ_i, THEN $\sigma_{k,i}$, THEN $\sigma_{k-1,i}$, ..., THEN $\sigma_{2,i}$, THEN $\sigma_{1,i}$.

EDGE TYPES

INTRA-LAYER EDGES : EDGES (u,v) WITH $u,v \in V_i$ FOR SOME $i \in \{0,\dots,p\}$

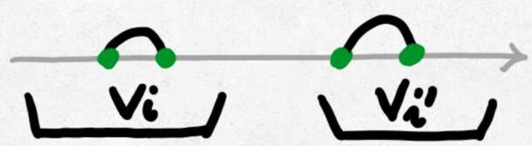
$\Rightarrow$ $\ell \cdot \log_2 n$ STACKS

EVEN INTER-LAYER EDGES : EDGES (u,v) WITH $u \in V_i$, $v \in V_{i+1}$ FOR SOME $i \in \{0,2,4,\dots\}$ $\Rightarrow$ $2\ell \cdot \log_2 n$ STACKS

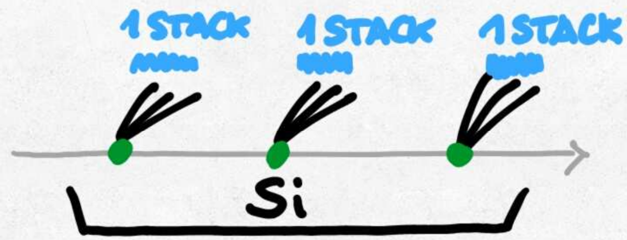
ODD INTER-LAYER EDGES : EDGES (u,v) WITH $u \in V_i$, $v \in V_{i+1}$ FOR SOME $i \in \{1,3,5,\dots\}$ $\Rightarrow$ $2\ell \cdot \log_2 n$ STACKS

INTRA-LAYER EDGES

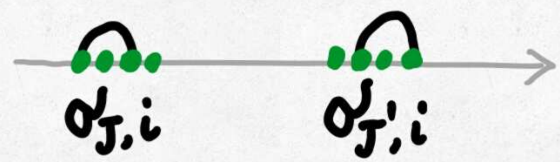
- AN EDGE BETWEEN TWO VERTICES IN A LAYER V_i DOES NOT CROSS AN EDGE IN A LAYER $V_{i'}$ WITH $i' \neq i$.
  $\Rightarrow$ IT SUFFICES TO LOOK AT A SINGLE LAYER V_i



- ℓ STACKS ACCOMMODATE ALL THE EDGES INCIDENT TO VERTICES IN S_i $\Rightarrow$ IT REMAINS TO ACCOMMODATE THE EDGES IN GRAPHS $G_1, G_2, \dots, G_k$



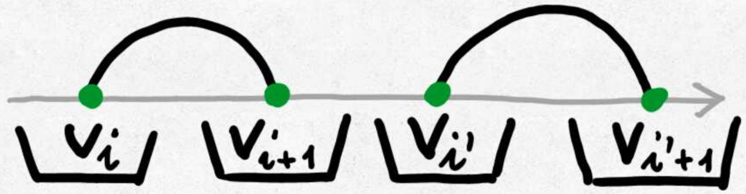
- AN EDGE IN G_J DOES NOT CROSS AN EDGE IN $G_{J'}$ IF $J' \neq J$ $\Rightarrow$ CAN USE THE SAME STACKS FOR ALL G_JS.



THE NUMBER $f_{INTRA}(n)$ OF STACKS USED TO ACCOMMODATE ALL THE INTRA-LAYER EDGES IS AT MOST $\ell + f_{INTRA}(n/2) \leq \ell \cdot \log_2 n$

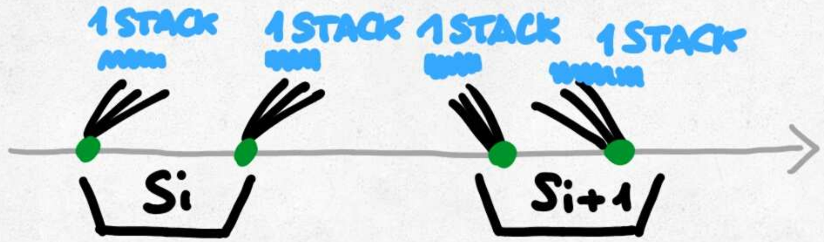
EVEN INTER-LAYER EDGES (1)

- AN EDGE BETWEEN A VERTEX IN A LAYER V_i AND A VERTEX IN A LAYER V_{i+1} DOES NOT CROSS AN EDGE BETWEEN A VERTEX IN A LAYER $V_{i'}$ AND A VERTEX IN A LAYER $V_{i'+1}$ WITH $i' \neq i$, GIVEN THAT $|i-i'| \geq 2$.



  $\Rightarrow$ IT SUFFICES TO LOOK AT A SINGLE PAIR OF LAYERS V_i AND V_{i+1}.

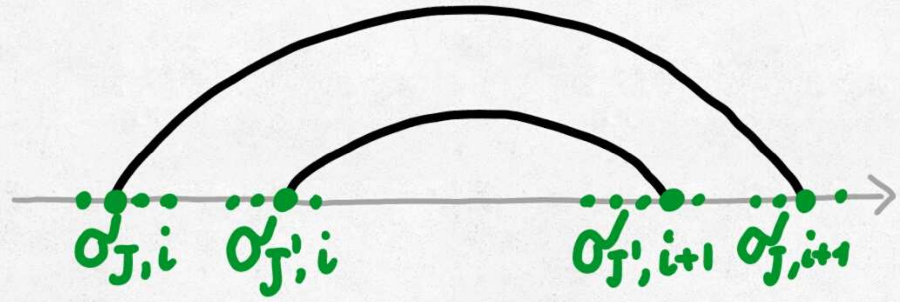
- 2ℓ STACKS ACCOMMODATE ALL THE EDGES INCIDENT TO VERTICES IN S_i AND ALL THE EDGES INCIDENT TO VERTICES IN S_{i+1}



  $\Rightarrow$ IT REMAINS TO ACCOMMODATE THE EDGES IN GRAPHS $G_1, G_2, \dots, G_k$

EVEN INTER-LAYER EDGES (2)

- AN EDGE IN G_J DOES NOT CROSS AN EDGE IN $G_{J'}$ IF $J \neq J'$ $\Rightarrow$ CAN USE THE SAME STACKS FOR ALL G_J S.



THE NUMBER $f_{\text{INTER-ODD}}(n)$ OF STACKS USED TO ACCOMMODATE ALL THE ODD INTER-LAYER EDGES IS AT MOST $2\ell + f_{\text{INTER-ODD}}(n/2) \leq 2\ell \cdot \log_2 n$

SUMMING UP

$\ell \cdot \log_2 n$ STACKS FOR THE INTRA-LAYER EDGES

$2\ell \cdot \log_2 n$ STACKS FOR THE EVEN INTRA-LAYER EDGES

$2\ell \cdot \log_2 n$ STACKS FOR THE ODD INTRA-LAYER EDGES (DISCUSSION ANALOGOUS TO THE ONE FOR THE EVEN INTRA-LAYER EDGES).

THEOREM IF AN n-VERTEX GRAPH ADMITS A LAYERED ℓ-SEPARATOR, THEN ITS STACK NUMBER IS AT MOST $5\ell \cdot \log_2 n$.

RESEARCH DIRECTIONS

- FIND NEW APPLICATIONS FOR LAYERED SEPARATORS.
- IS THE STACK NUMBER OF k-PLANAR GRAPHS CONSTANT (FOR CONSTANT k)?