Monotone Simultaneous Embeddings of Paths in $\mathbb{R}^d$

David Bremner, Olivier Devillers, Marc Glisse, Sylvain Lazard,
Giuseppe Liotta, Tamara Mchedlidze, Sue Whitesides, Stephen Wismath

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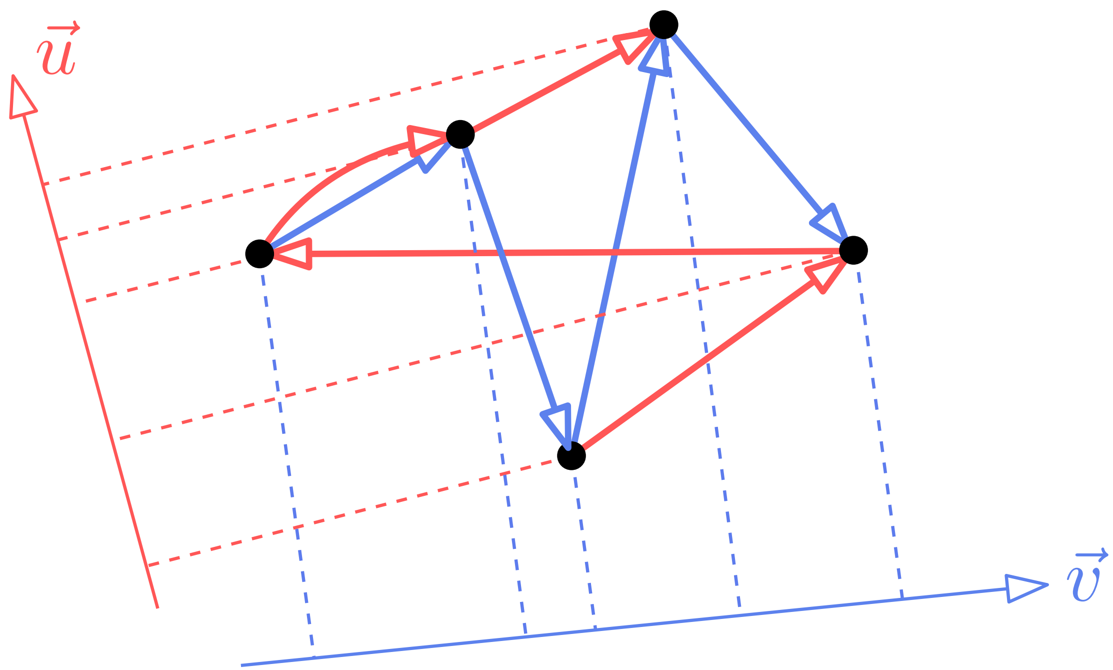
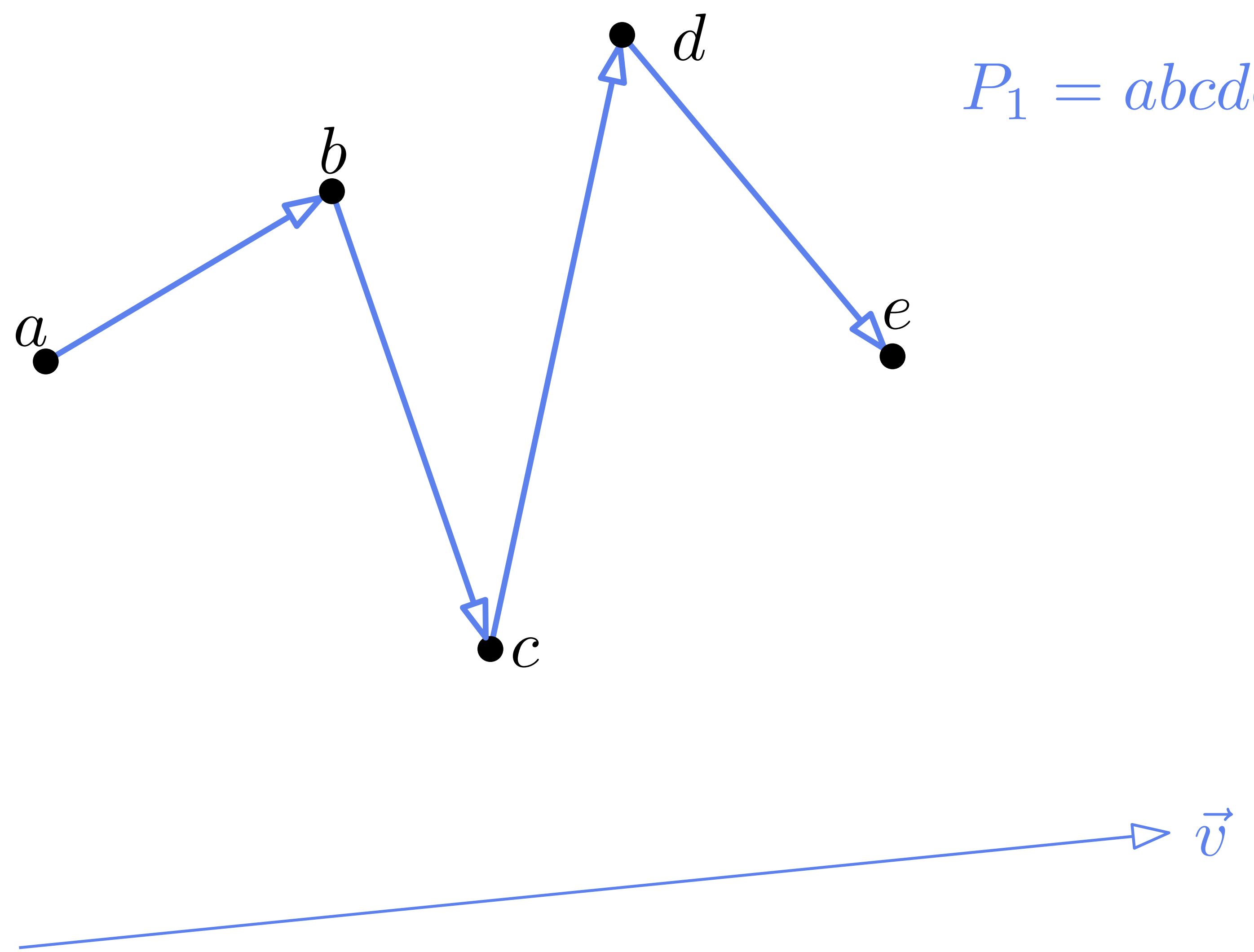


Monotone Simultaneous Embedding

A path is **monotone** wrt $\vec{v}$



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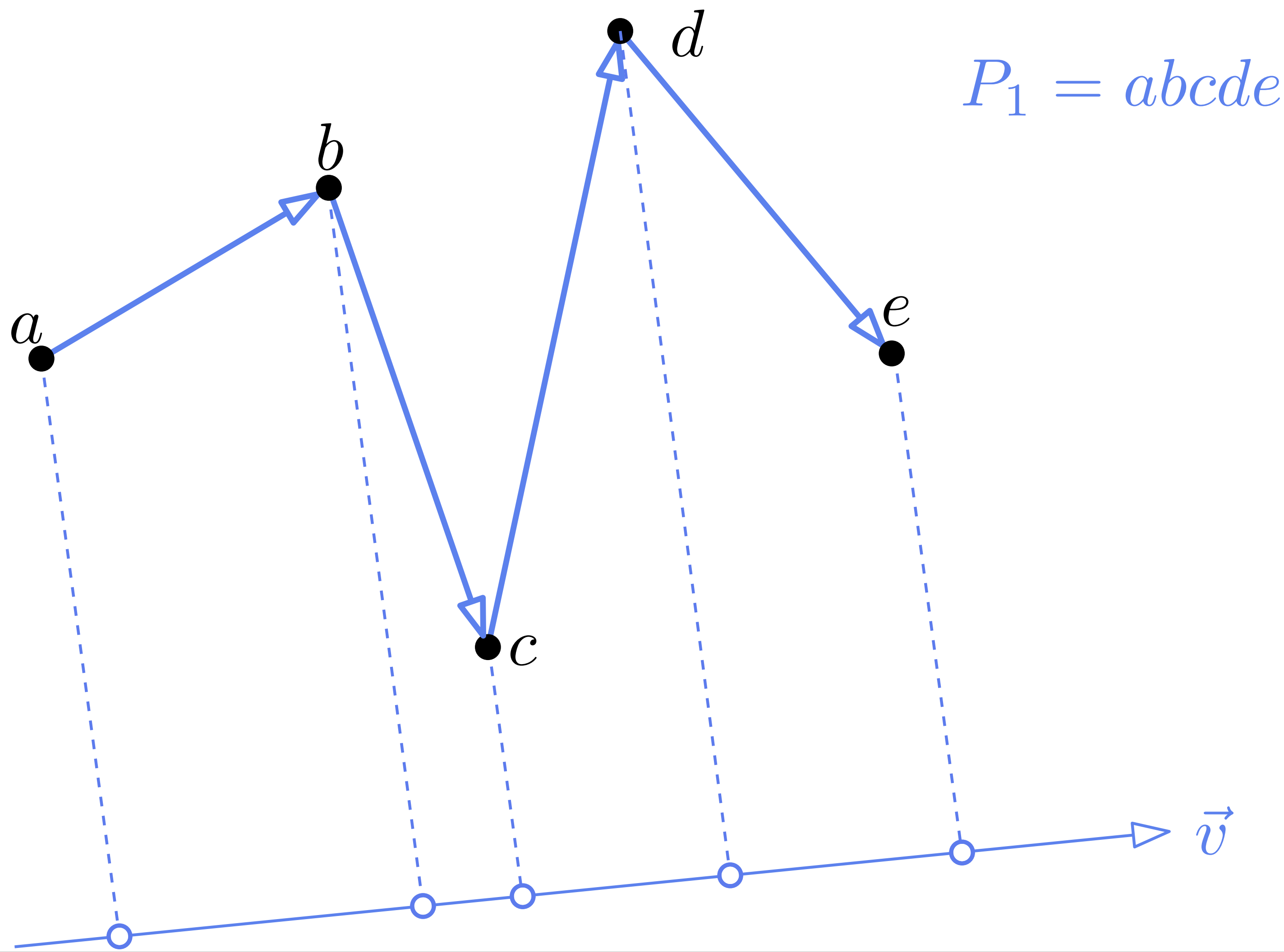

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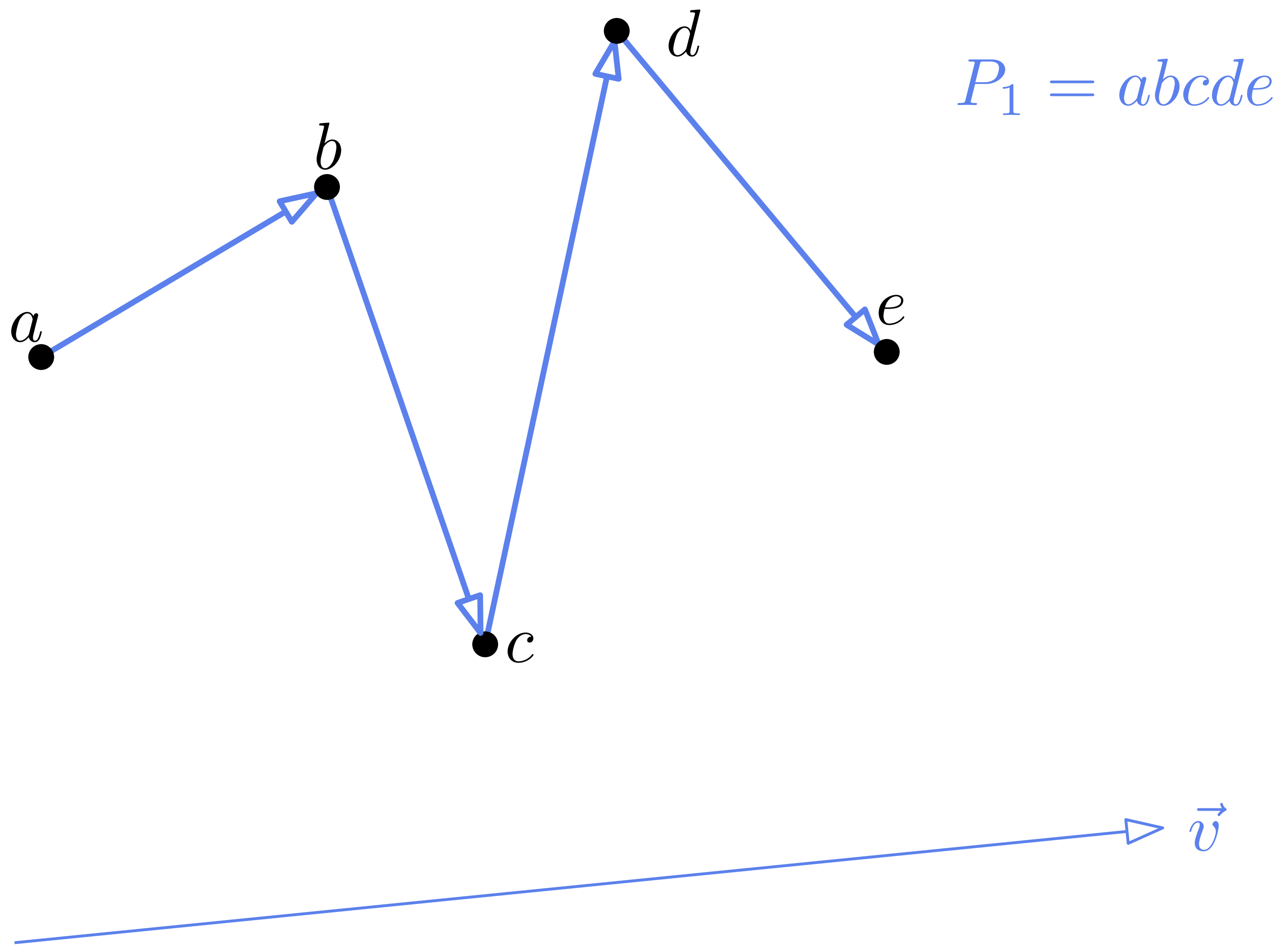

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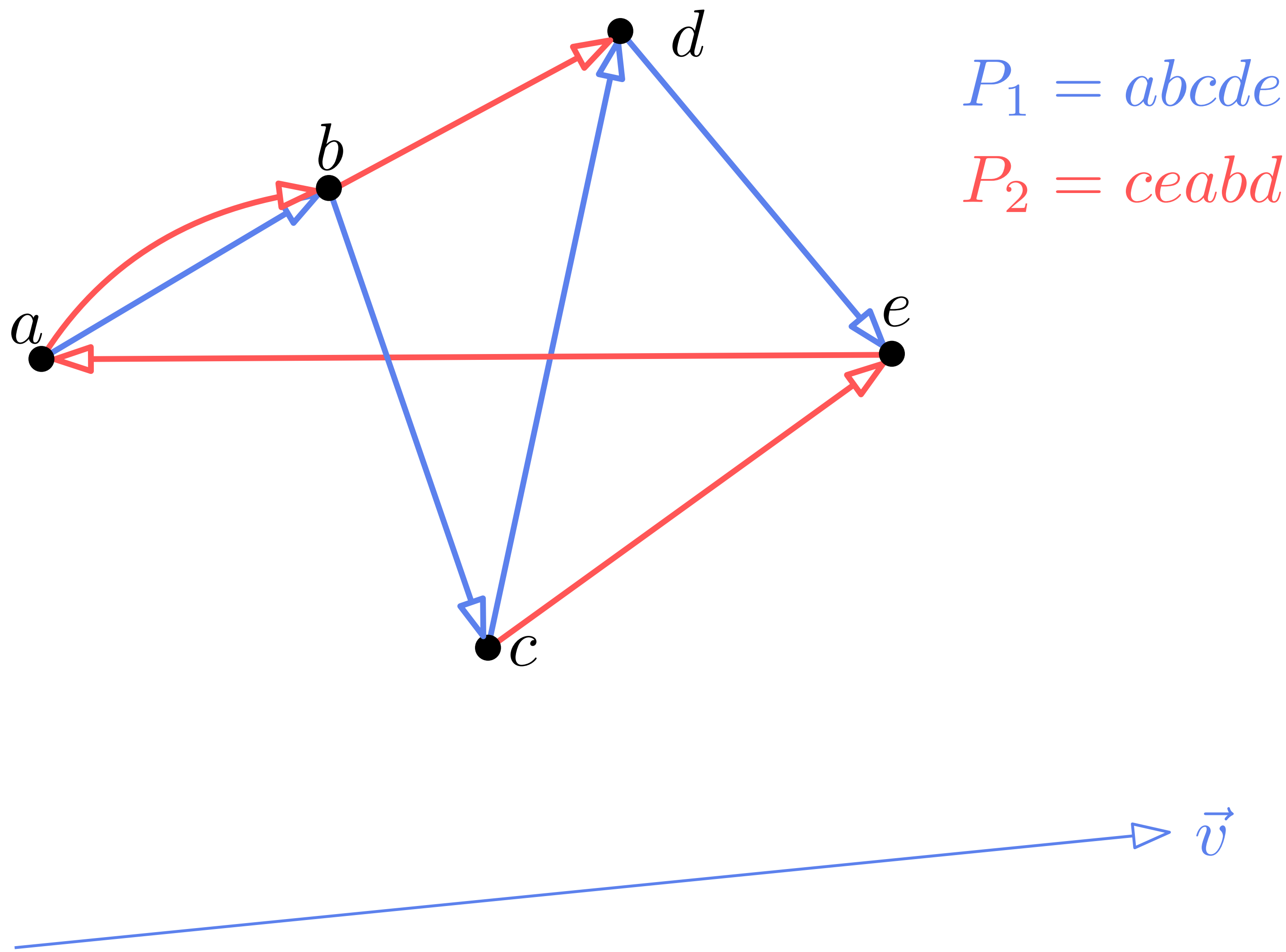

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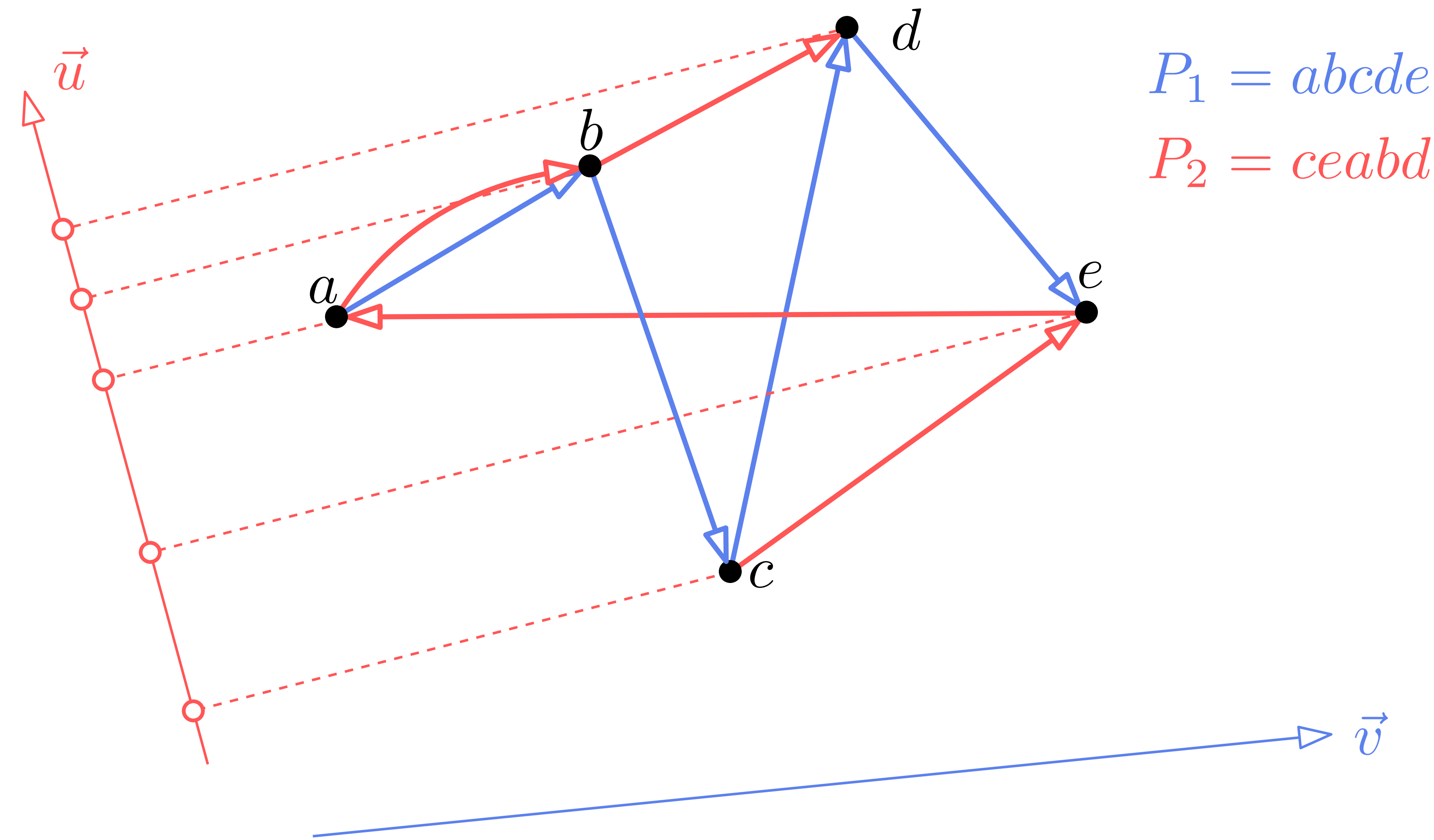

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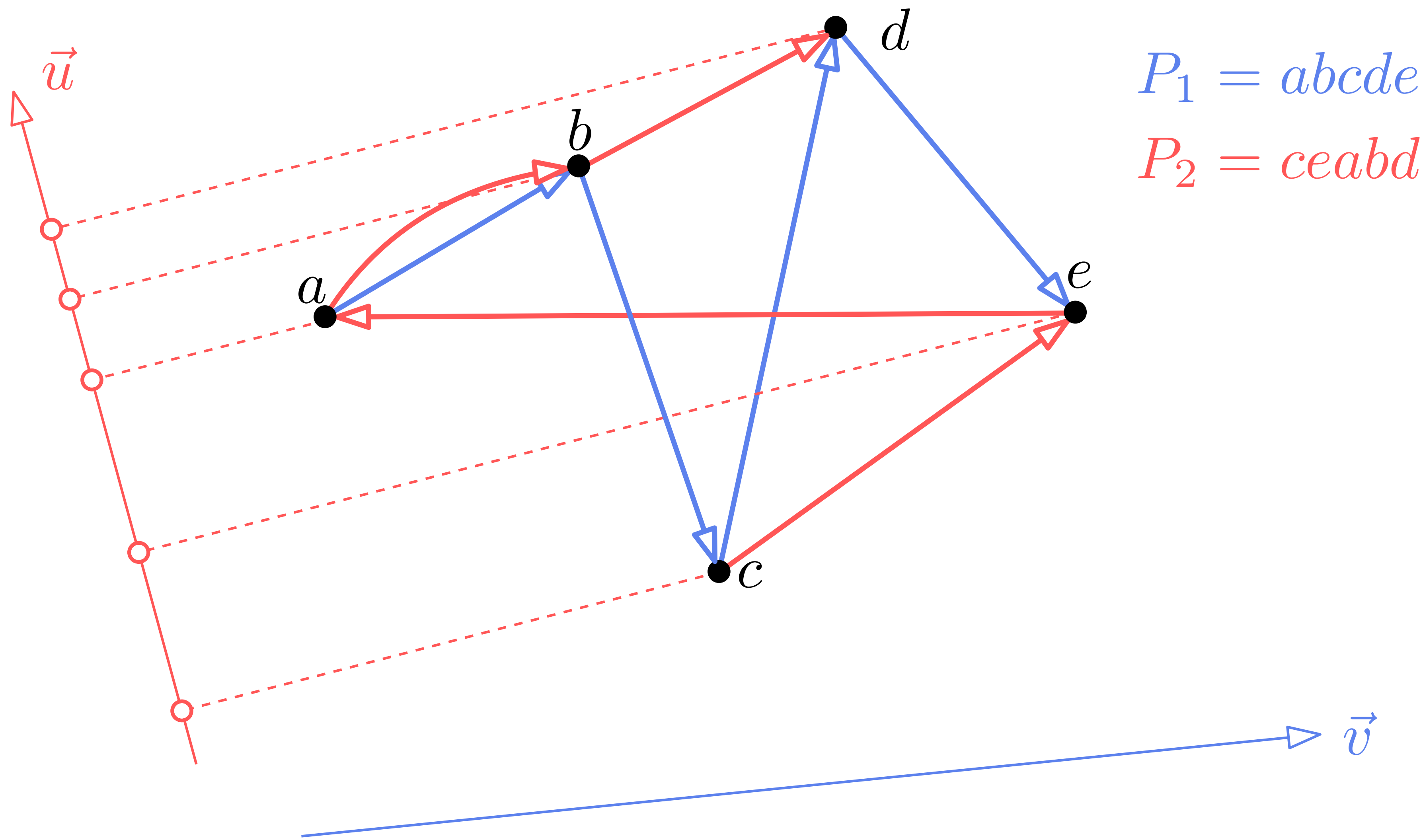
Monotone simultaneous embedding (MSE) of P_1 and P_2



Previous work: 2-D

- Two directed paths on the same vertex set always admit a monotone simultaneous embedding

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- The decision question without given directions of monotonicity for 3 paths is polynomial, and is unknown for ≥ 4 paths

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Definition: duality transform

$$p = (a, b, c) \qquad \mapsto \qquad p^* : z = ax + by - c$$

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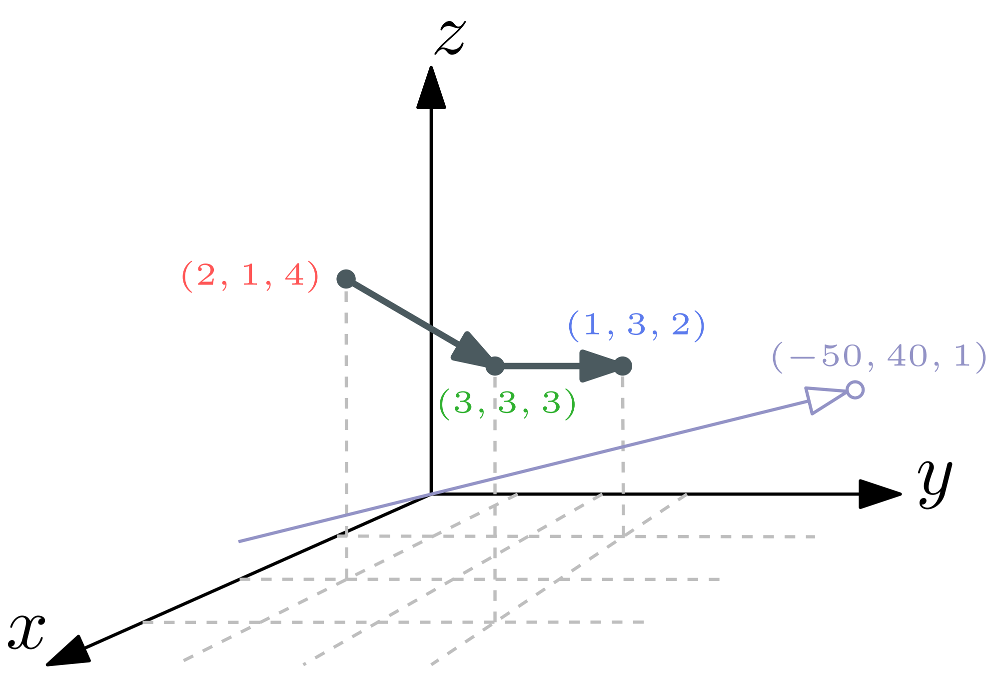

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$z = 2x + y - 4$

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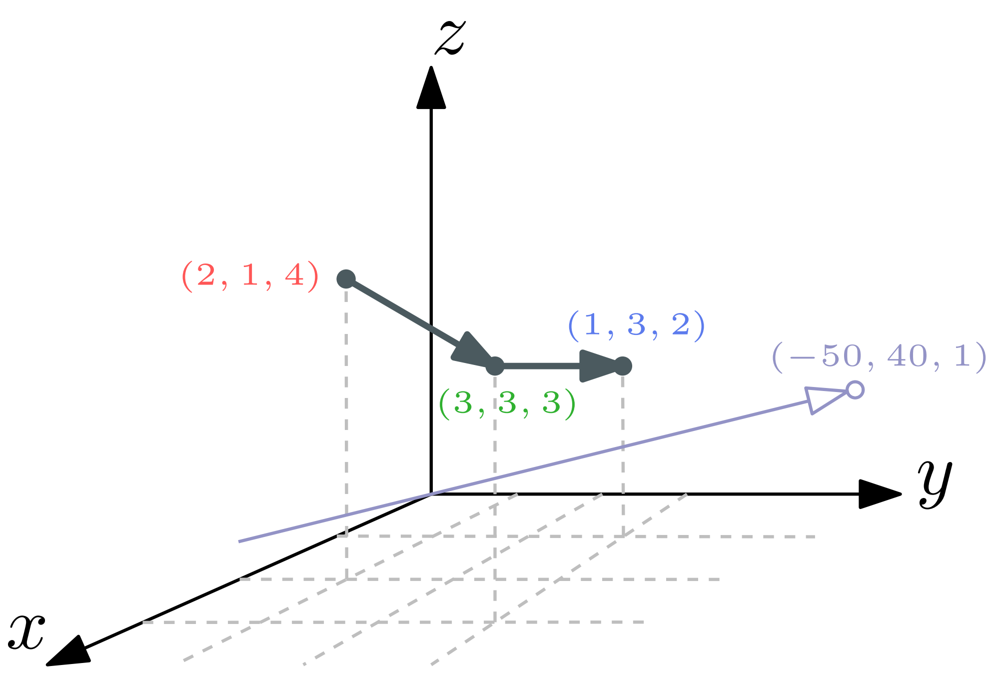

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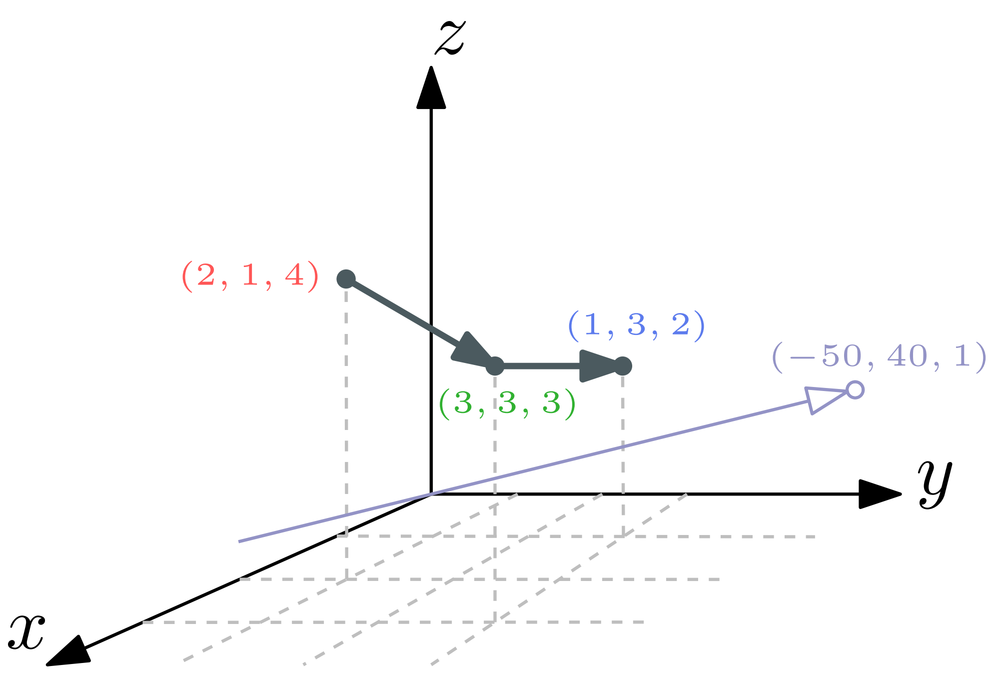
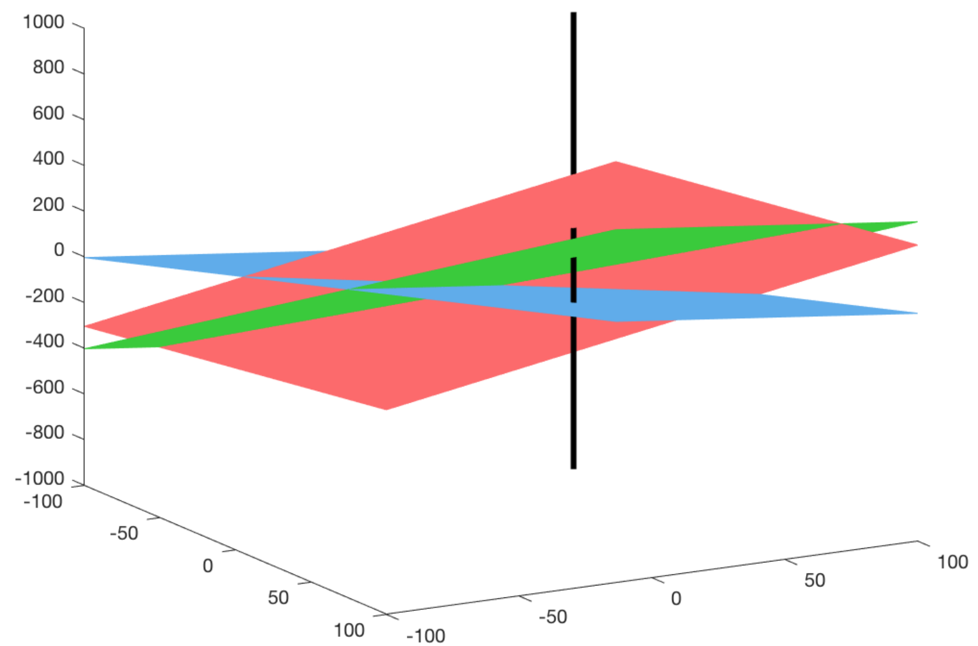
- vector $(-50, 40, 1)$
- orthogonal plane $-50x + 40y + z = d$
- dual point: $(50, -40, -d)$





MSE in the Dual Plane

- Let $\Pi = \{P_1, \ldots, P_k\}$ be a set of directed paths on the same vertex set
- We denote by $\hat{\Pi}$ the set of directed paths that is produced from Π by reversing some of its paths

Lemma (duality lemma): Let Π be a set of paths on the same set of vertices $1 \ldots n$. Π admits a MSE in 3-dimensional space iff there exist

- $\hat{\Pi} = \{P_1, \ldots, P_k\}$
- a set of planes $H_1, \ldots, H_n$, and
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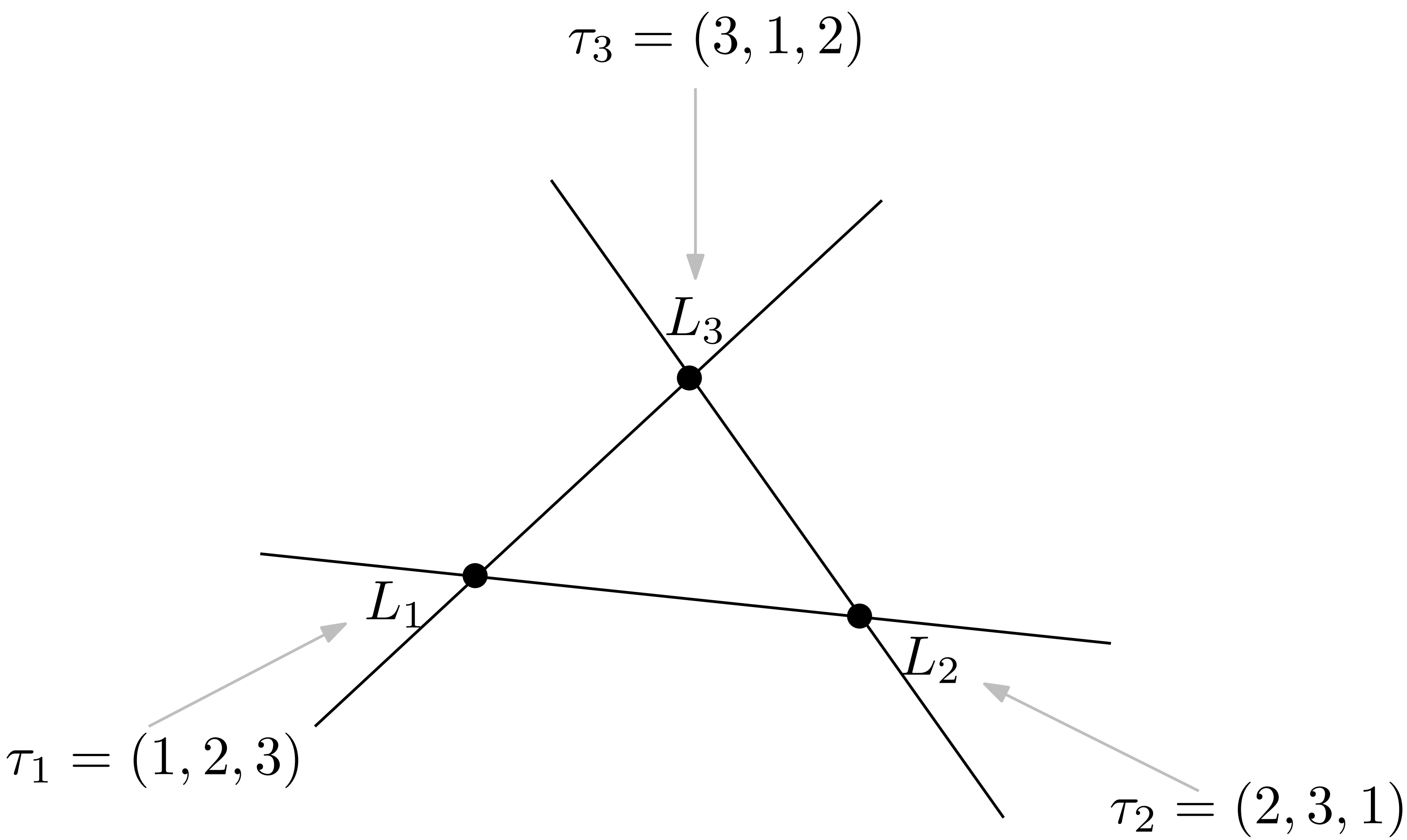
the order in which L_i intersects $H_1, \ldots, H_n$ is the same as the order of the vertices $1, \ldots, n$ in P_i, $1 \leq i \leq k$.

Next

Theorem 1: There exists 4 paths P_1, P_2, P_3, P_4 on 5 vertices such that there exist no vertical lines L_1, L_2, L_3, L_4 and planes H_1, H_2, H_3, H_4, H_5 so that the order in which L_i crosses the planes is the same in which the vertices appear in P_i, $i = 1, 2, 3, 4$.

Theorem 2: There exists a set of 4 paths on 40 vertices that do not admit a MSE in 3D.

Proof of Theorem 1



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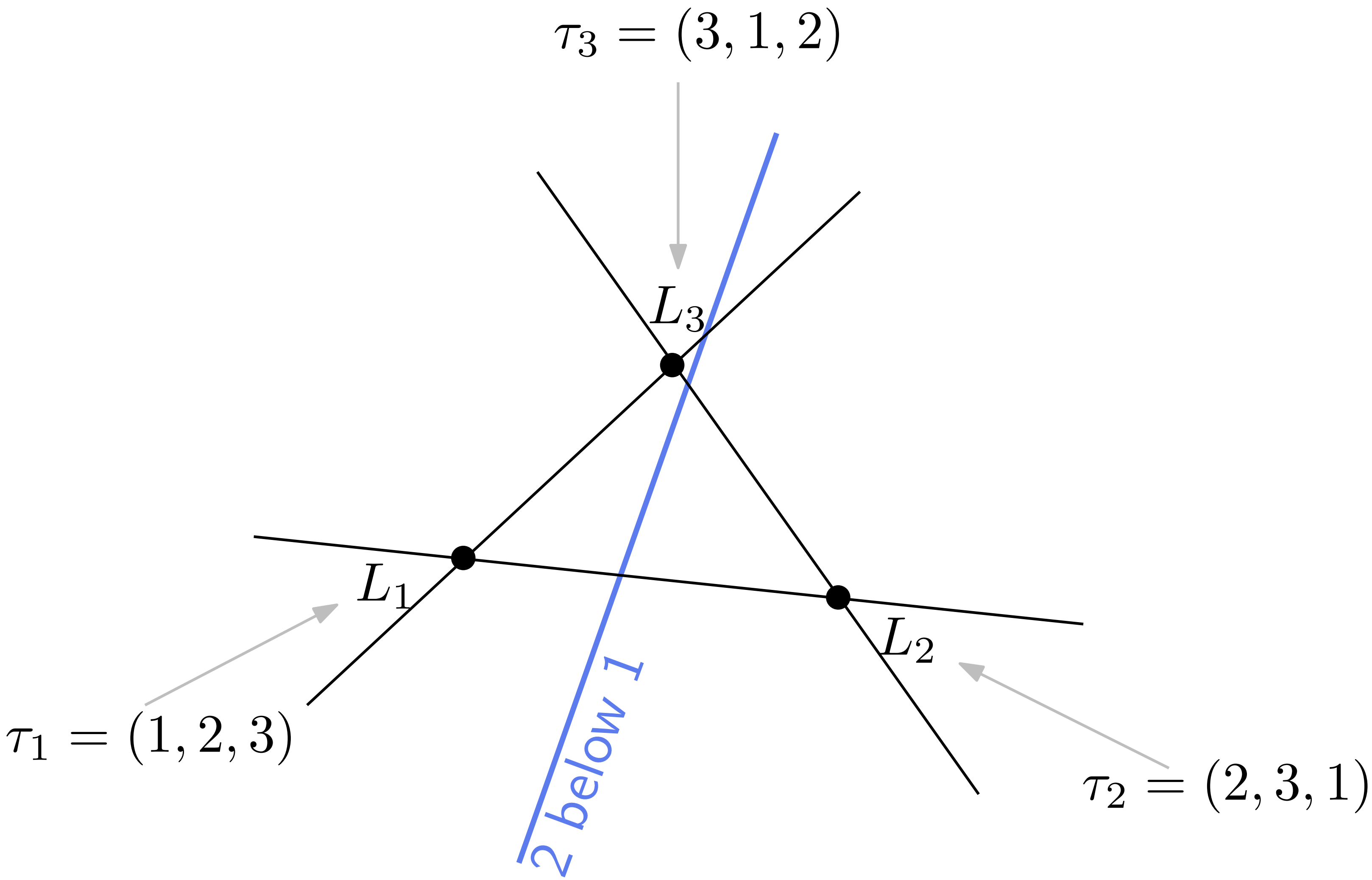

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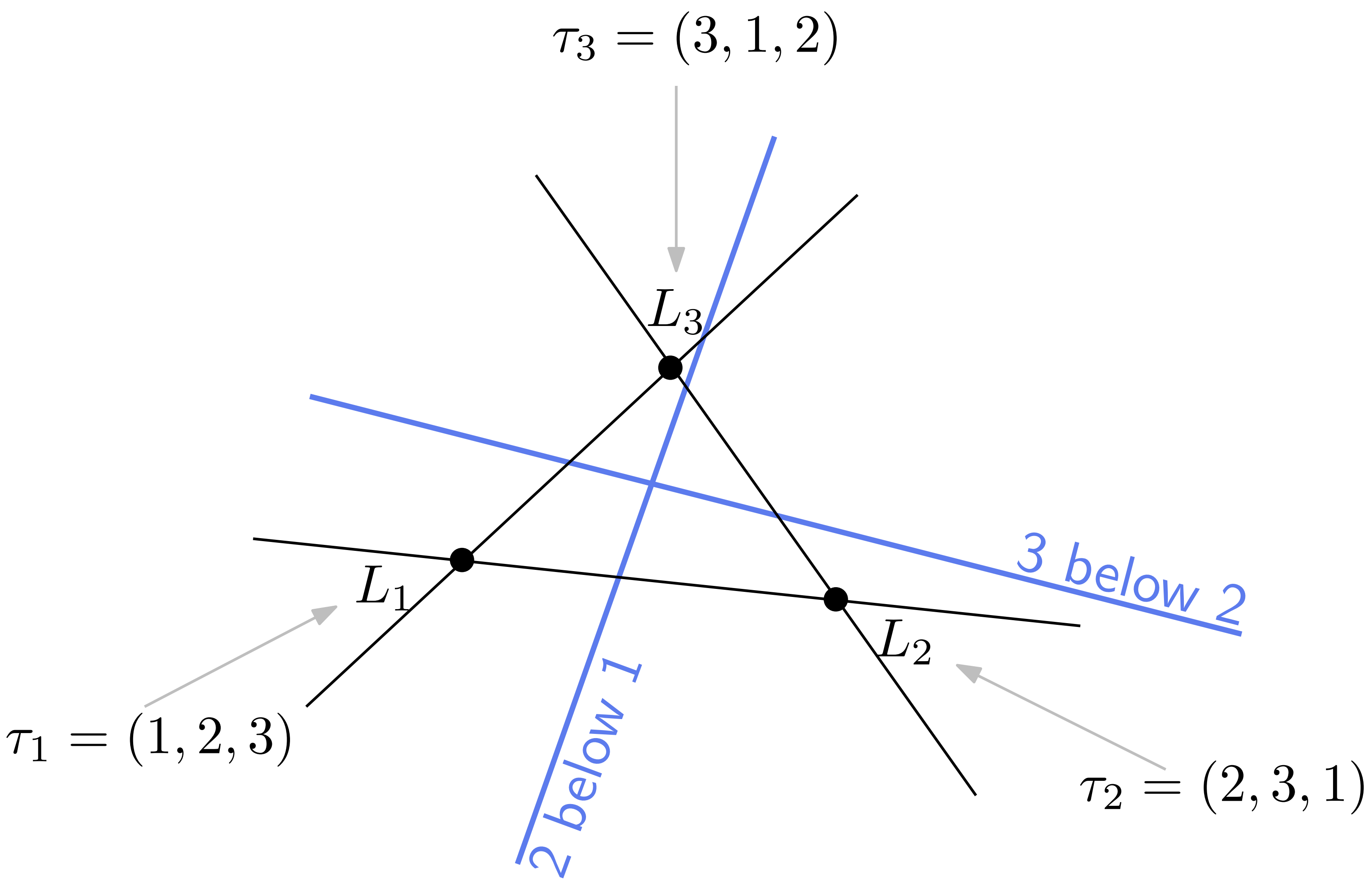

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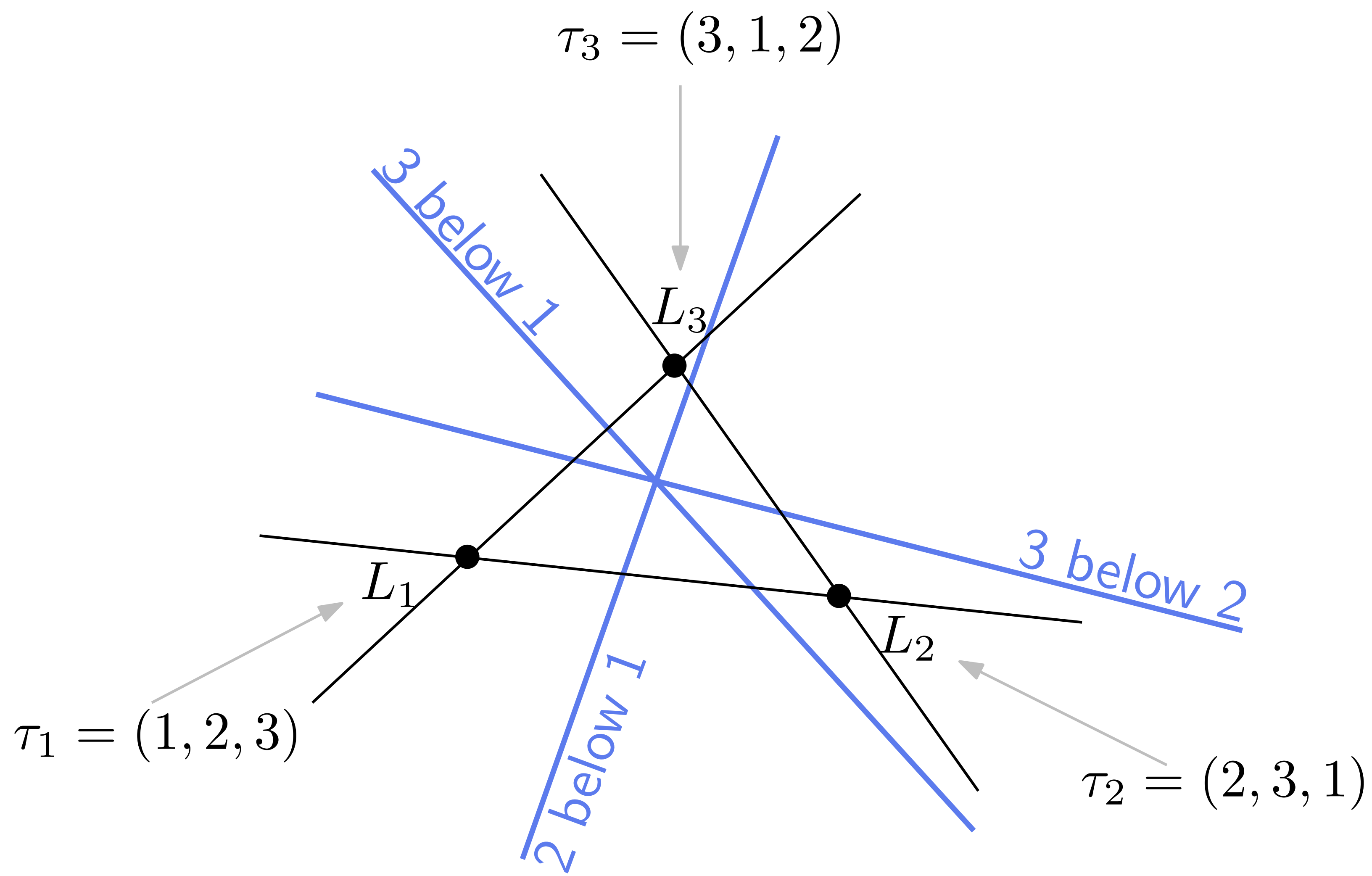

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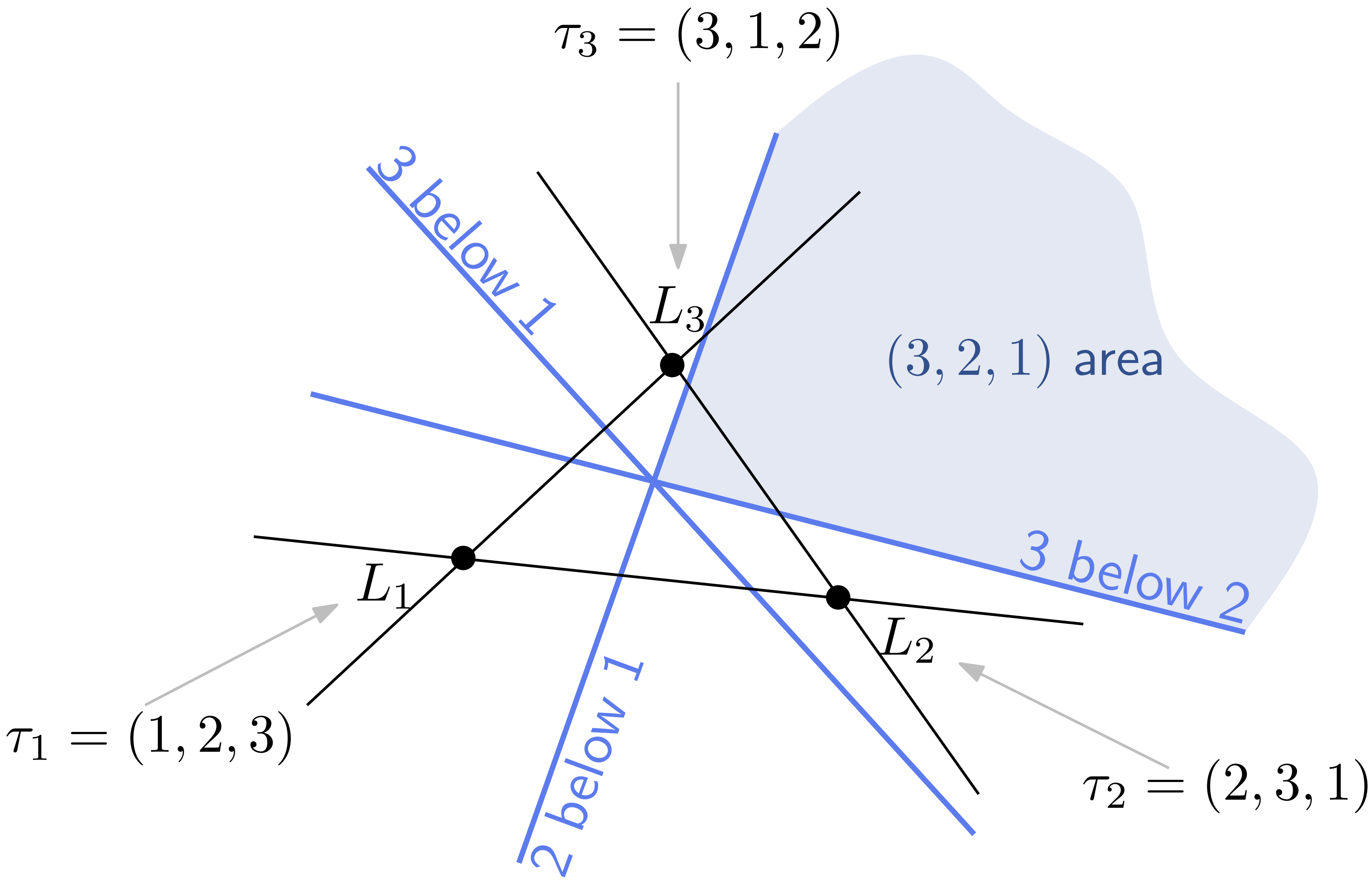

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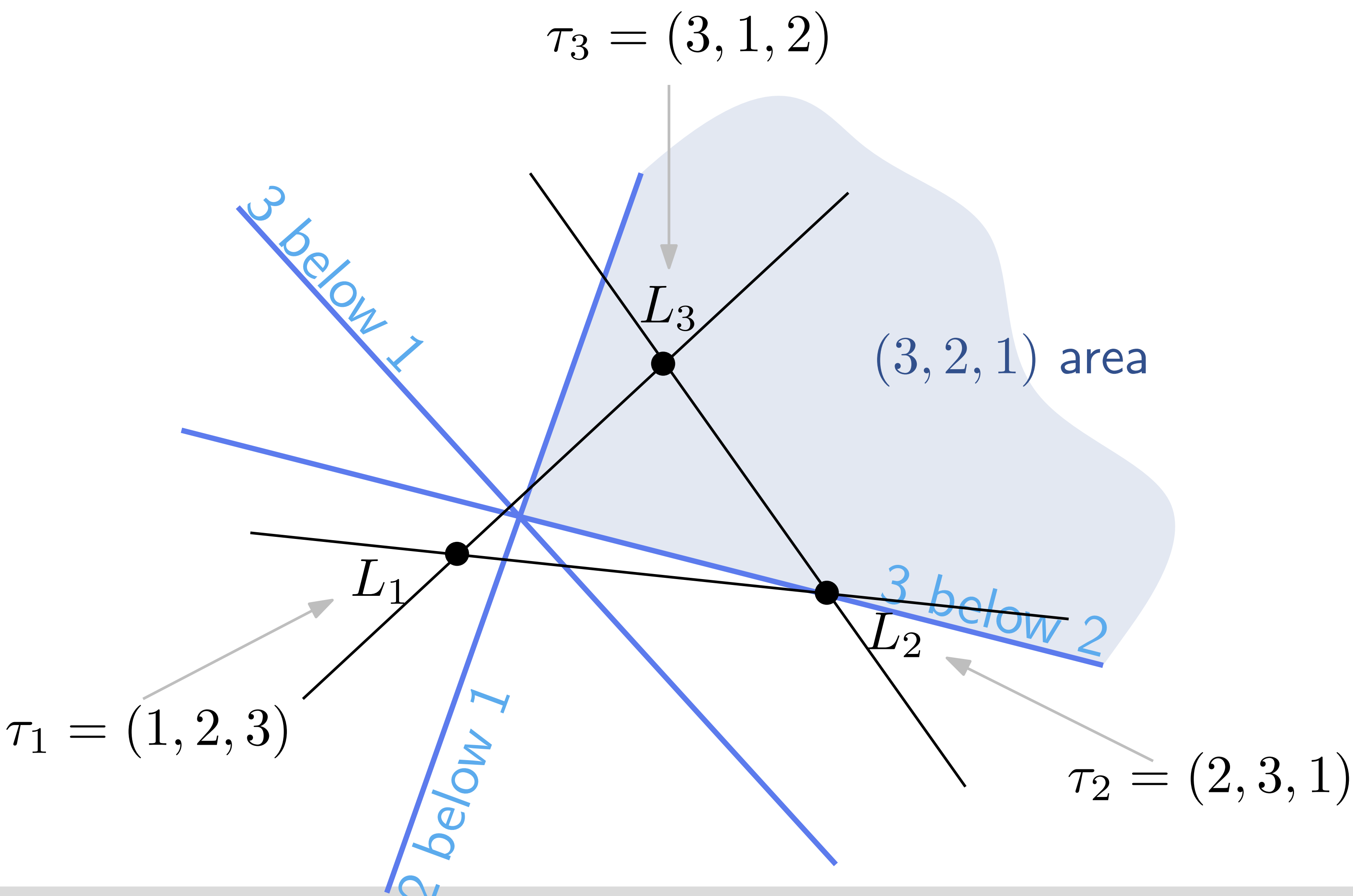

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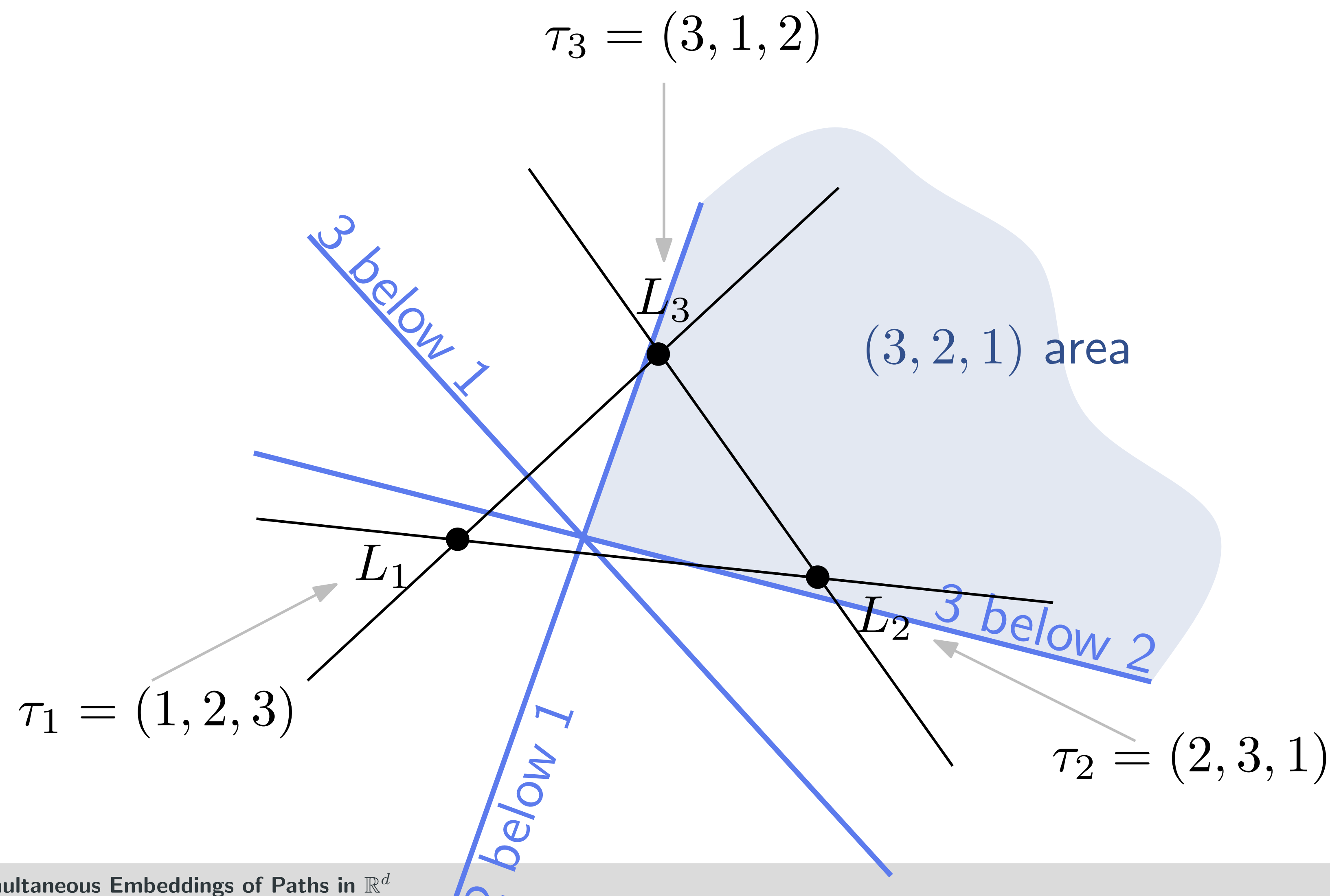

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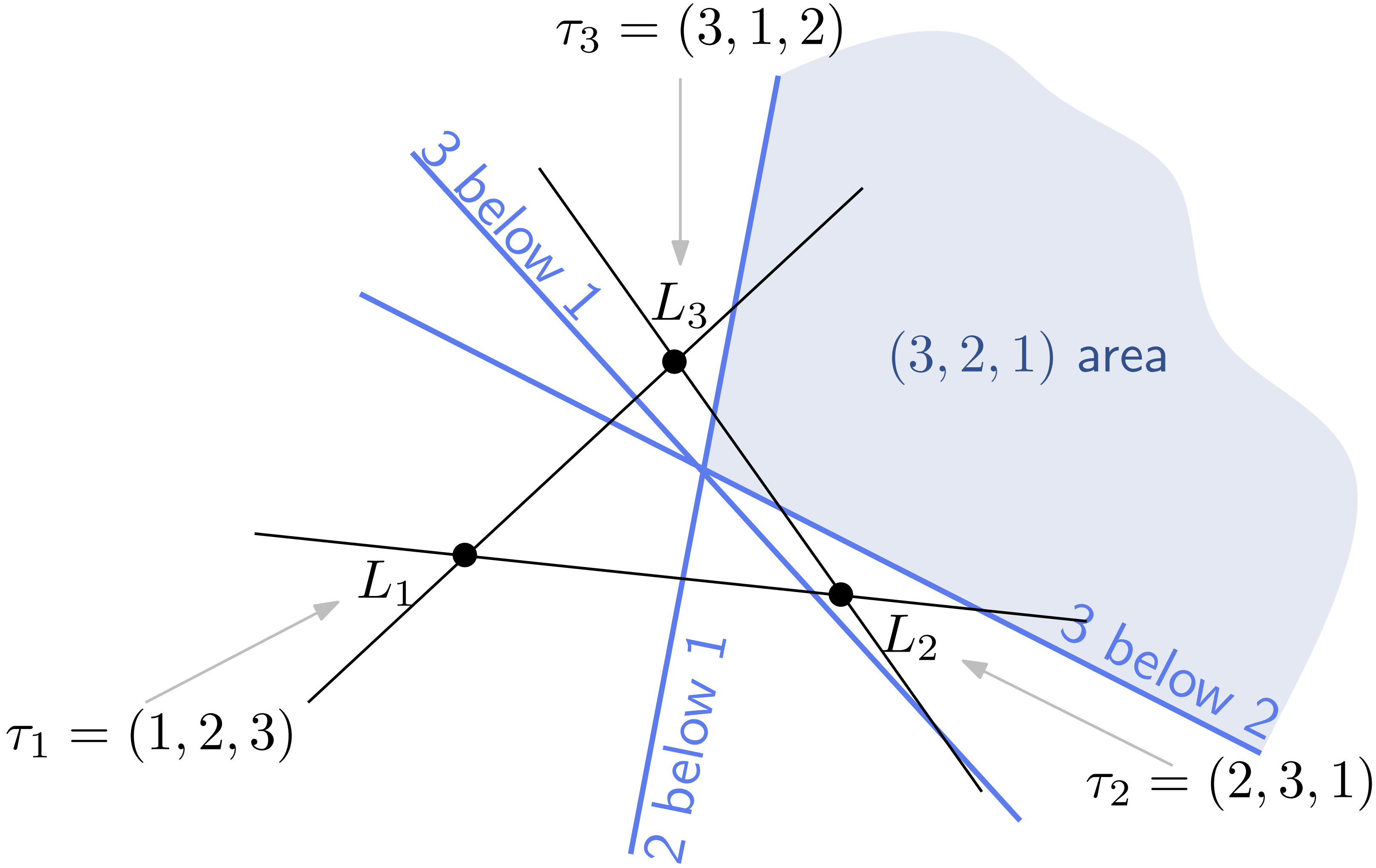

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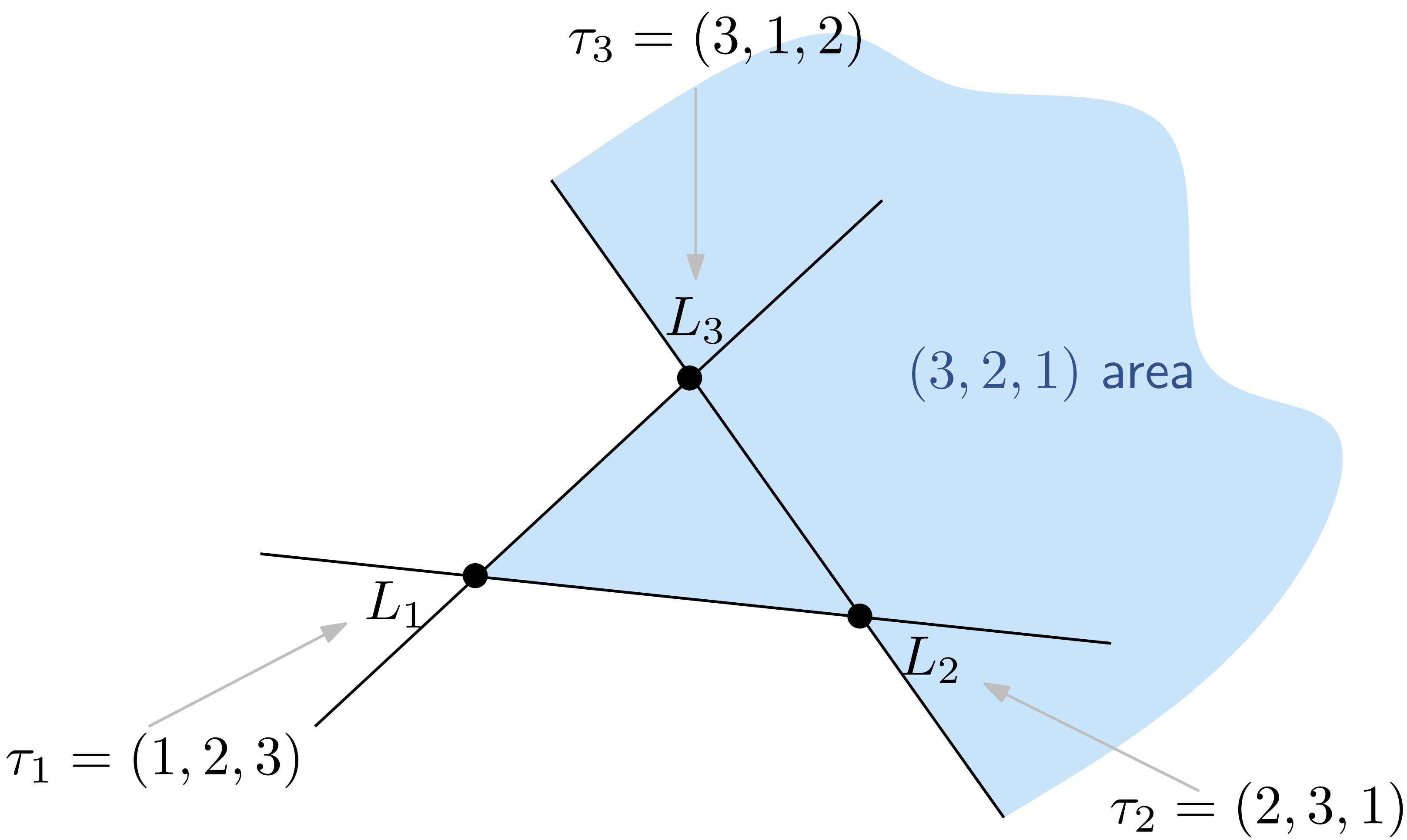

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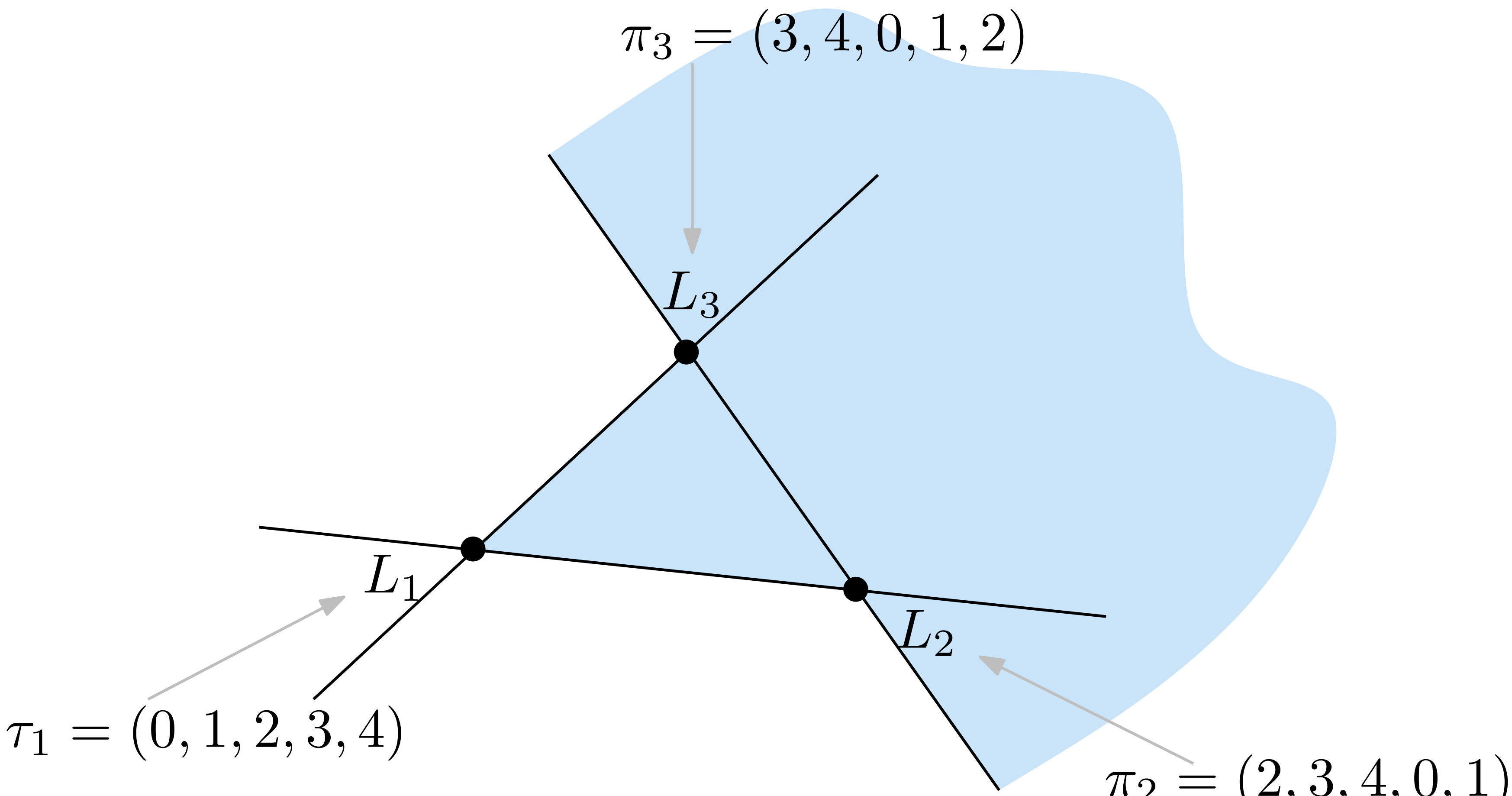

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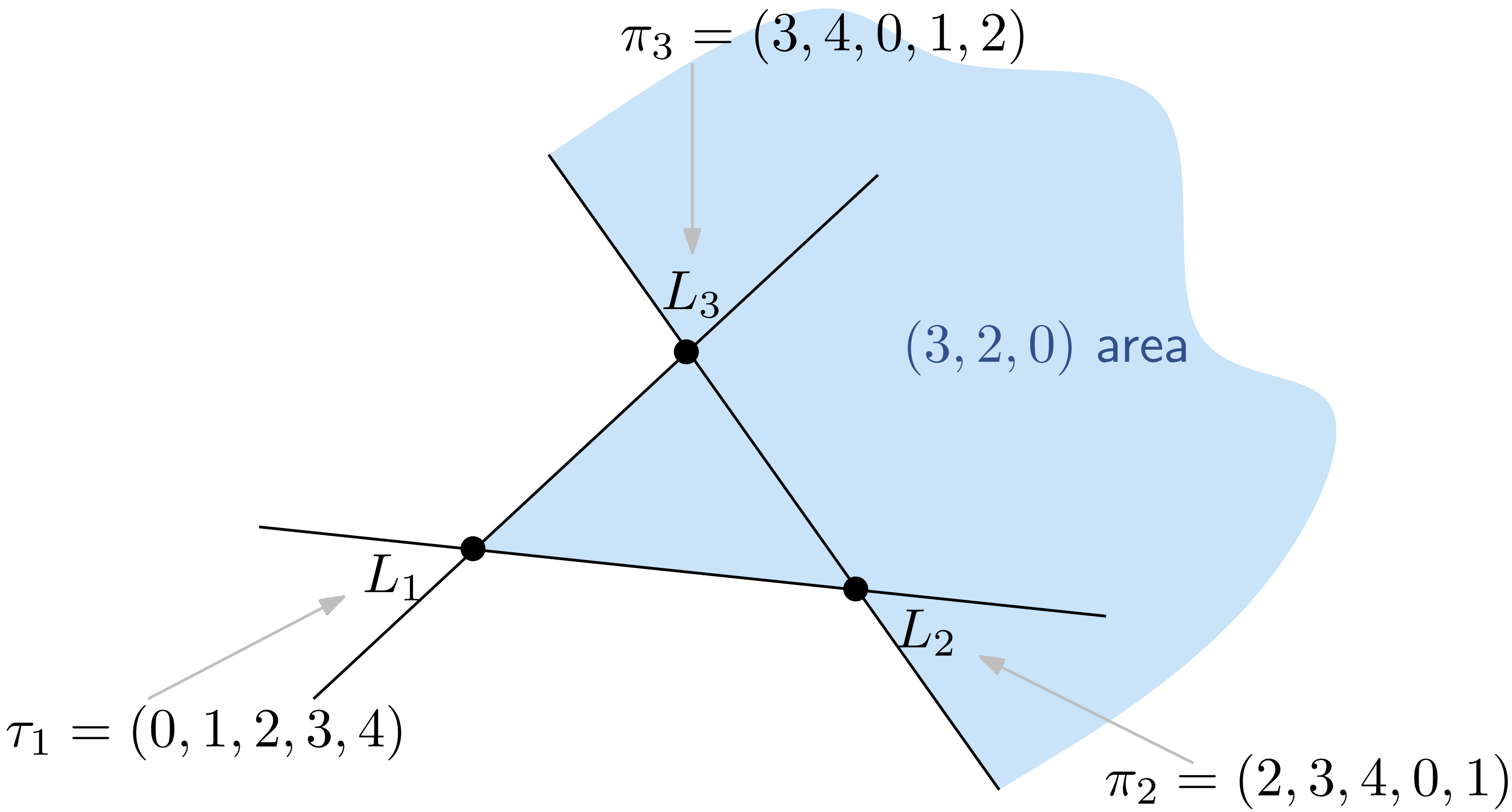

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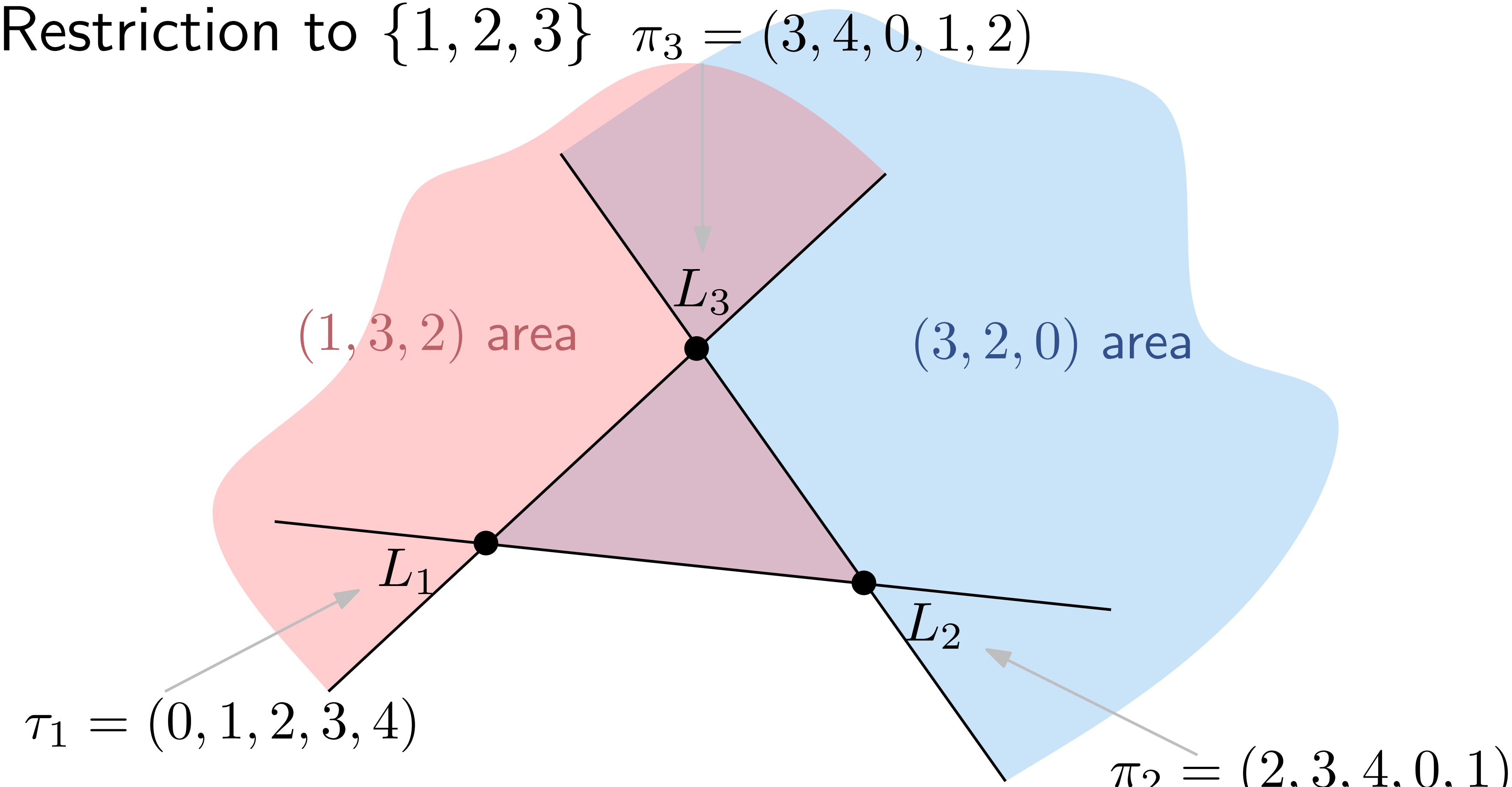

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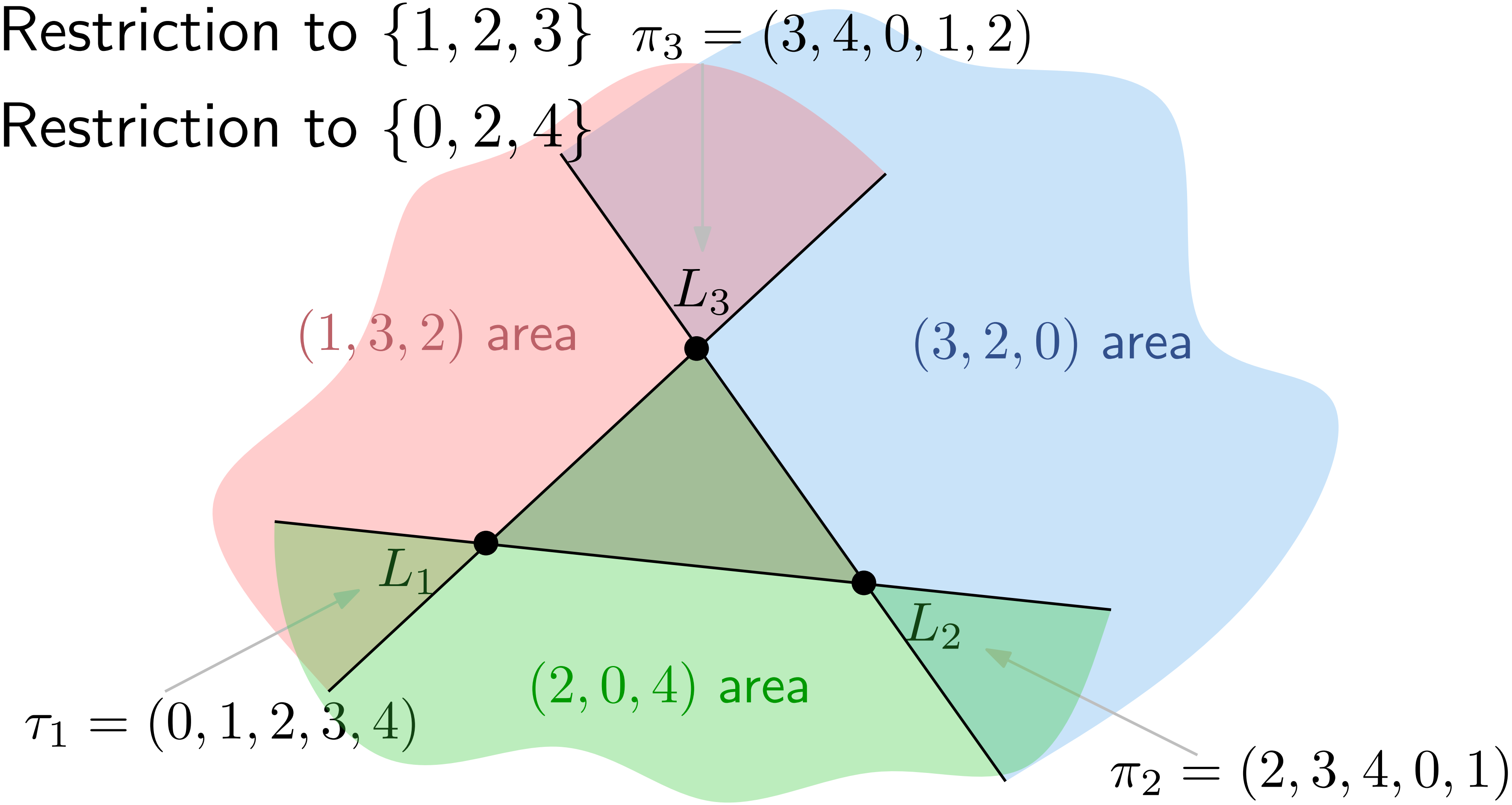

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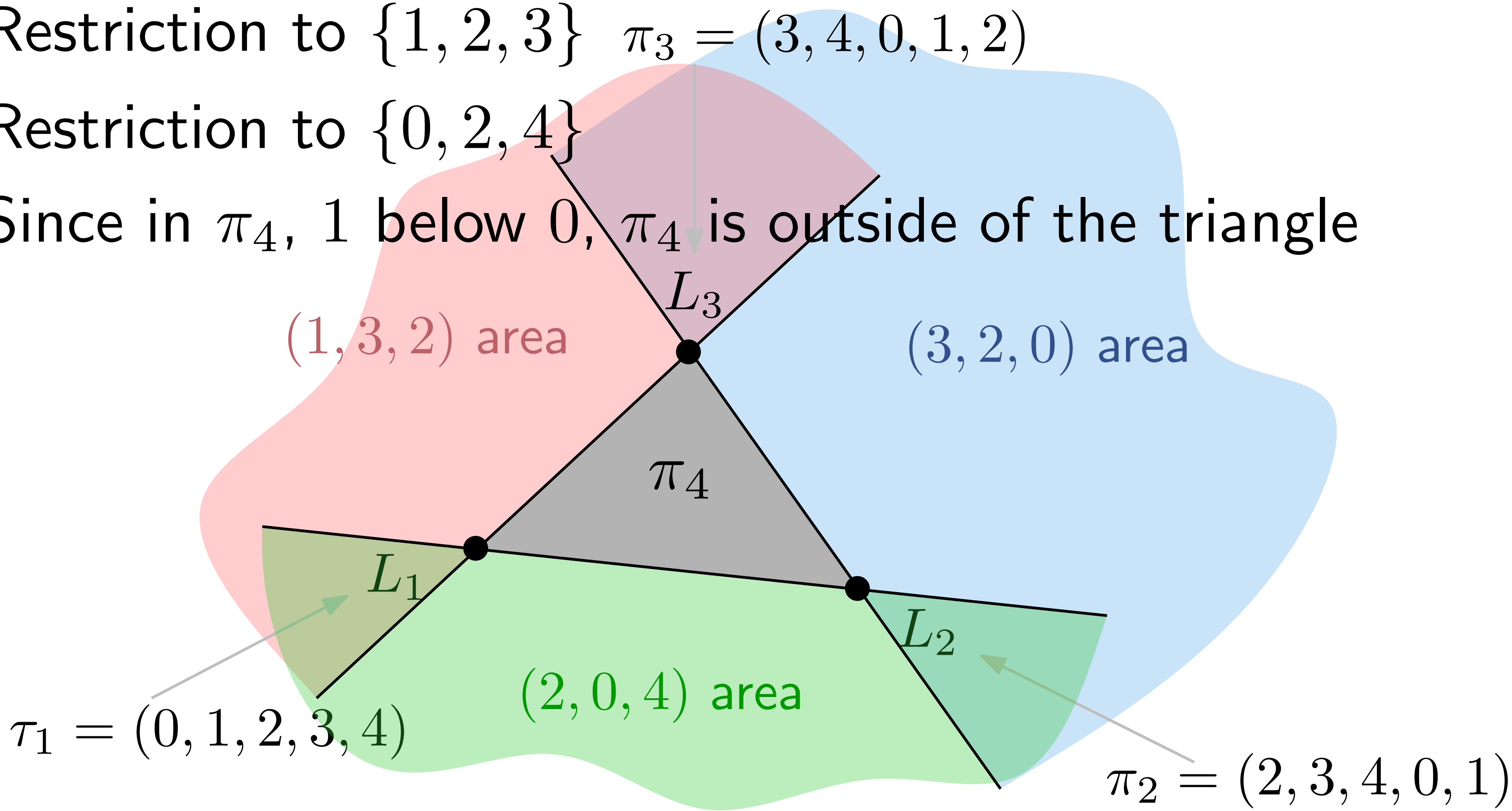
- Since in π_4, 1 below 0, π_4 is outside of the triangle



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$20, 21, 22, 23, 24,$	$22, 23, 24, 20, 21,$	$22, 21, 20, 24, 23,$	$21, 23, 22, 20, 24,$
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$40, 41, 42, 43, 44,$	$41, 40, 44, 43, 42,$	$43, 44, 40, 41, 42,$	$41, 43, 42, 40, 44,$
$50, 51, 52, 53, 54,$	$51, 50, 54, 53, 52,$	$53, 54, 50, 51, 52,$	$54, 50, 52, 53, 51,$
$60, 61, 62, 63, 64,$	$61, 60, 64, 63, 62,$	$62, 61, 60, 64, 63,$	$61, 63, 62, 60, 64,$
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Thank You!